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Cardinality and autoscaling: Revisiting the content and format of the approximate number system

Jacob Beck
Department of Philosophy
Centre for Vision Research
York University

Sam Clarke
Department of Philosophy
Department of Psychology
University of Southern California

Abstract: This chapter considers the content and format of approximate number representations. In previous work, we have defended the orthodox view that these representations represent *numbers* in an *analog* format. The present treatment defends and refines these suggestions, discussing recently advocated alternatives according to which approximate number representations represent cardinalities or numerousness instead of numbers, and a novel account of their format dubbed "autoscaling" by its chief proponent, C.R. Gallistel.

1. The Approximate Number System

Take a quick glance at Figure 1. Without scrutinizing the image too closely, notice how easy it is to tell – just by looking – that Panel B contains a larger *number* of dots than Panel A. You might feel that you simply *see* this, with all the immediacy with which you *see* the dots to be of a different color. There is no need to laboriously count them.

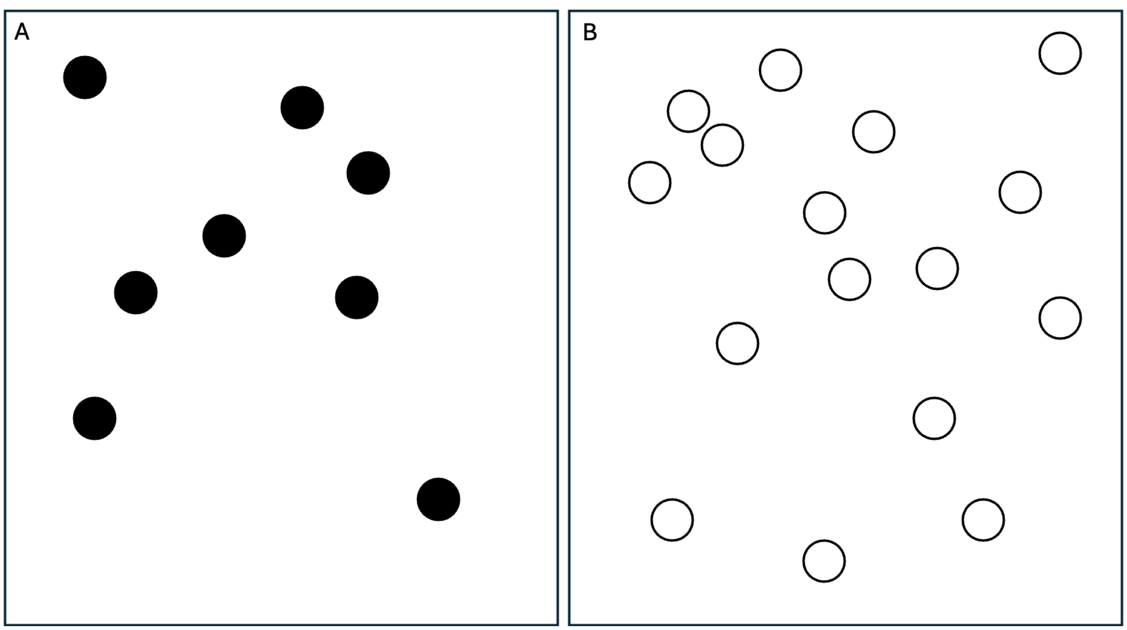

Figure 1: Which panel contains a larger number of dots?

According to an orthodox view in cognitive science, this is unsurprising. Just as we possess visual mechanisms which represent the size or color of the dots, we are likewise equipped with a 'number sense' or *approximate number system* (ANS). As its name suggests, this is a psychological *system*, which functions to represent the *number* of items in perceived collections. But unlike our mature counting abilities, the system only enumerates collections *approximately*, with its operations conforming to Weber's Law. So, much as the accuracy with which we perceptually discriminate the weight of two buckets, or height of two lampposts, is predicted by the ratio among those heights or weights, rather than their absolute difference, ANS performance is predicted by *the numerical ratio* between collections: the further from 1:1, the better. 8 dots are easier to discriminate from 7 dots than 9 dots, even though 8 differs from 7 and 9 by the same absolute amount.

In past work, we've explored two features of the ANS: its *content*, or *what* it represents; and its *format*, or *how* it represents. With respect to content, we've argued that the ANS represents numbers—in fact, rational numbers such as 7 and 1.5 (Clarke & Beck 2021a, 2021b). With respect to format, we've contended that ANS representations are analog, like a mercury thermometer (Beck 2015; 2019; Clarke 2022a; 2023). Our aim in this chapter is to clarify, develop, and extend these positions by addressing some challenges and open questions. We focus on two issues.

1. Does the ANS really represent *numbers*? Or does it instead represent primitive analogs of number, such as cardinalities, numerosities, or numerousness, as some thinkers maintain (Bermúdez 2021; Dutilh Novaes & Dos Santos 2021; Opfer et al. 2021; Dos Santos 2022; Samuels & Snyder 2024)?

2. Some researchers have advanced a new theory of the format of ANS representations called *autoscaling* (Gallistel 2011, 2017; Odic et al. 2024). But what is autoscaling? And how does it relate to the idea that ANS representations have an analog format?

Each issue could be the subject of its own chapter, so our ambitions must be correspondingly modest. But we aim to at least give readers a sense of what is at stake and to advance the discussion, taking on the issue of *content* in Section 2 and the issue of *format* in Section 3. These sections are largely independent, so readers primarily interested in autoscaling may safely skip ahead.

2. Content

Holding that the ANS represents numbers conflicts with a familiar line of thinking: Numeracy and mathematics reflect the height of the human intellect. Numbers only became objects of human awareness when mathematicians began studying them. Since evidence for an ANS is found in fish, rats, pigeons, monkeys, and infants, it couldn't really represent numbers.

Two versions of this worry should be distinguished. According to the first, the ANS represents not number, but non-numerical magnitudes that correlate with number, such as surface area. In earlier work (Clarke & Beck 2021a §4), we argued that this skeptical hypothesis is difficult to square with phenomena like the connectedness illusion and cross-modal studies. We thus set this version of the worry aside here (though see Park, this volume, for a new version of the worry according to which the ANS represents "normalized local contrast at multiple spatial scales").

The second version of the worry grants that the ANS represents something *numerical* but denies that it represents numbers. Instead, it represents primitive analogs of number. Several candidates for these primitive analogs have been proposed, including that they are quantical dimensions (Núñez 2017), pure magnitudes (Burge 2010; Gross et al. 2021), cardinalities (Dutilh Novaes & Dos Santos 2021; Opfer et al, 2021; Bermudez 2021; Samuels & Snyder 2024), or numerosities/numerousness (Dos Santos 2022). Here we'll focus on the final two suggestions—that the ANS represents *cardinalities* (§2.1) and that it represents *numerosities* or *numerousness* (§2.2). Our discussion is intended to refine and extend our earlier critique of these proposals (Clarke & Beck 2021a, 2021b).

2.1 Cardinalities

We want to approach the hypothesis that the ANS represents cardinalities indirectly, by first introducing a distinction between two concepts of *number*. We introduce these two concepts to mark a notional distinction. Whether this distinction is merely notional or corresponds to distinct entities is a controversial question in the philosophy of mathematics that we do not attempt to settle here. Nevertheless, we hope to highlight the options available and to consider how they relate to our suggestion that the ANS represents numbers.

First, there is what we'll call *the mathematicians' concept of number*. This is the concept at play when mathematicians study numbers and attribute properties, such as being prime or being even, to them. It is exemplified by sentences like

(1) The number seven is prime.

In (1), 'seven' expresses the mathematicians' concept of a number.[1] Again, what such numbers are is controversial. At the very least, they are abstract. You cannot point to the mathematicians' number seven or locate it in space or time. They are also generally agreed to be objects—entities that can bear properties. Sometimes, though not always, they are also taken to be particulars, which cannot be instantiated.

Second, there is what we'll call *the quotidian concept of number*. This is the concept we deploy when we count or attribute numerical properties to collections, including concrete pluralities (collections that are located in space and time).[2] It is exemplified by sentences like

(2) There are seven apples on the table.

Taking the surface grammar of (2) at face value, 'seven' attributes a property to a concrete plurality (the apples on the table). Thus, seven is not being treated as an object. On its face, 'seven' appears to have a different sense in (2) than in (1).

The ANS is, perhaps, most naturally taken to represent quotidian numbers, at least in paradigm cases. That's because its primary home is in perception, where it attributes quotidian numbers to concrete pluralities. Thus, your ANS might enable you to see some dots as being (approximately) seven in number, or to hear some tones as being (approximately) ten in number. *Prima facie*, such applications are much closer to the concept of number exemplified by (2) than that exemplified by (1). In fact, it's not clear how perception *could* put one in touch with numbers in the mathematicians' sense. Perception connects perceivers to concrete entities (via perceptual reference) and to the instantiation of properties (via attribution). But mathematicians' numbers, as typically conceived, are abstract objects.

As noted, this is a notional distinction. One might deny that it picks out two different types of number. For example, one might attempt to reduce mathematicians' numbers to quotidian numbers by analyzing numbers in the mathematicians' sense as reified properties of collections (Kim 1981;

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[1] Similarly, Moltmann (2013) says that 'seven' in (1) refers to a "pure number."

[2] For a discussion of what collections are in this context, see Florio this volume.

Maddy 1990; Yi 2018).[3] On this view, there would be little daylight between saying that the ANS represents quotidian numbers and saying that it represents numbers *simpliciter*.

But some maintain that the distinction is not only notional but real: that there are two different kinds of numerical entities (Moltmann 2013; Snyder 2017; Samuels & Snyder 2024). In that case, space emerges for holding that the ANS represents quotidian numbers without representing anything like mathematicians' numbers. This matters because some researchers use the term 'cardinality' to refer to what we have been calling quotidian numbers and reserve the term 'number' for what we've been calling mathematicians' numbers. Thus, they take 'seven' in (1) to refer to (what they would call) a *number* and 'seven' in (2) to attribute (what they would call) a *cardinality* (Opfer et al. 2021; Samuels & Snyder 2024).

In this sense of 'cardinality', we are happy to allow that the ANS represents cardinalities. It's just important to distinguish this sense of 'cardinality' from others in circulation. In set theory, for example, the term is often used to denote the size of a set. When sets are interpreted as abstract objects, this set-theoretic usage aligns more closely with the mathematicians' concept of number. But other uses aside, whether one uses the term 'cardinality' or 'quotidian number' strikes us as a verbal matter. We thus continue to think that there is nothing wrong with the slogan "the number sense represents numbers," even if it's quotidian numbers that are at issue.

There are, however, two important complications. First, the thesis that the ANS represents cardinalities/quotidian numbers does not yet deliver our earlier thesis that the ANS represents rational numbers (Clarke & Beck 2021a). But all we meant by this is that the ANS represents not only whole numbers, but also ratios among whole numbers and fractions. For example, the ANS might represent not only that there are (approximately) fifteen blue dots and (approximately) five yellow dots, but also that (approximately) three-quarters of the dots are blue (Qu et al. 2024). And we see no reason to deny that quotidian numbers might enter into ratios/fractions in this way.

Second, while the primary home of the ANS is in perception where it is applied to concrete pluralities, it can also be deployed in cognition. For example, when subjects are tasked with approximately subtracting two numbers presented symbolically (as Arabic numerals), the intraparietal sulcus, which is associated with the ANS, is active (Piazza et al. 2007). Likewise, when subjects are tasked with identifying the larger of two numbers, presented as Arabic numerals, response times are predicted by the ratio between the numbers discriminated, suggesting the ANS's involvement (Moyer & Landauer 1967). Since the subtraction or comparison of two numbers that aren't associated with any concrete pluralities seems to involve the mathematicians' concept of number, this suggests that the ANS is not limited to representing quotidian numbers. It can also come to represent mathematicians' numbers when it's embedded within a larger cognitive system. In saying this, we want to emphasize that we are not taking a stand on how these mathematicians' numbers should be analyzed—for example, whether they should be understood as reified properties of collections or as autonomous abstract objects. Our point is simply that any account of the content of the ANS, including an account that centers on cardinalities, should accommodate the fact that the ANS is not limited to operating in circumstances where concrete pluralities are perceived.

[3] Samuels & Snyder (2024, pp. 6–7) call this "mathematical empiricism."

2.2 Numerosities and Numerousness

Even bracketing the above, ANS researchers often remain hesitant to say that the ANS represents numbers. And although some say that it represents cardinalities, this is rare. Most commonly, they say that the ANS represents *numerosities*. In past work (Clarke & Beck 2021a, 2021b), we argued that this neologism reflects a double standard. No one says that we only represent shape-osity, distance-osity, or duration-osity. So why claim that the ANS represents numerosity instead of number? We also complained that it is opaque what a numerosity is supposed to be, and that researchers rarely say much by way of elucidation.

One possibility is that 'numerosity' should be understood as a synonym for 'cardinality' or 'quotidian number' (Opfer et al. 2021). If so, we have no substantive objection to its use when formulating hypotheses about ANS content, though we wonder why such a strange term should be introduced when 'cardinality' or 'number' would serve just as well. (One might object that 'number' is ambiguous since it could mean quotidian number or mathematicians' number. But 'distance' can similarly mean physical distance or mathematical distance; and yet no one proposes that the term 'distance-osity' is needed to curtail confusion. In both cases, context of use suffices to clarify the intended meaning.)

We suspect, however, that most researchers are not using 'numerosity' as a synonym for 'cardinality'. By way of example, we consider a recent proposal by Dos Santos (2022) that seeks to resurrect and clarify the concept of numerosity. Although we reject the proposal, we choose to discuss it because it has the rare and admirable virtue of saying what a numerosity is meant to be.

Dos Santos traces a three-way distinction between number, numerosity, and numerousness to a lost paper by the great 20th-century psychophysicist Stanley Stevens (1939/2006). On Dos Santos' interpretation, numerosity is cardinality; number is a scale for measuring numerosity; and numerousness is a sensation. At first glance, this distinction between numbers and numerosities might seem close to a version of the distinction between mathematicians' numbers and quotidian numbers. But matters aren't so simple.

For Dos Santos, numerosity is a relational property between an observer and an aggregate (concrete plurality) that emerges when the observer counts members of the aggregate. Thus, without an observer, there is no numerosity. Dos Santos thinks that this relationality is necessary to capture the fact that one cannot answer a how-many question without recourse to a sortal. For example, the very same aggregate can be described as 52 cards, four suits, or one deck. And on Dos Santos' view, it's the observer who is responsible for applying the sortal, making numerosity observer dependent. No observer, no counting; no counting, no sortal; no sortal, no numerosity.

We reject this entirely. While cardinalities (quotidian numbers) must be relativized to a sortal, rendering them relational, this does not make them observer dependent. A helpful comparison is weight. Although Dos Santos says that weight is an intrinsic, non-relational property (p. 1164), this is mistaken. Weight is a relational property since it's determined not only by an object's mass (an intrinsic property) but also by the gravitational field. A rock that weighs 60 pounds on Earth only weighs 10 pounds on the moon. But that doesn't mean weight is observer dependent. The rock has a weight of 60 pounds relative to Earth whether anyone is around to weigh it or not. Likewise, the apples on Sam's counter total seven in number relative to the sortal *apple* whether Sam or anyone else is around to count them or not. Whether something is an apple is observer independent. Apple trees existed in the Miocene, long before humans were around to cultivate them or count their fruits.

Even so, there is a fact of the matter about how many apples there were. It makes sense to wonder whether the number of apples on the planet decreased with the onset of the bipolar ice age.

Regardless, Dos Santos' considered opinion is not that the ANS represents numerosity. Rather, he thinks it represents numerousness, "the sensation produced by the exposure to a collection of discrete items" (p. 1161). Numerousness emerges when an observer perceives an aggregate and her perceptual system automatically applies a sortal (e.g. *Spelke object*). Thus, numerousness is also deemed to be observer dependent. When a collection is not observed, it has no numerousness.

Given that numerousness is a sensation for Dos Santos, he is surely right that it is observer dependent. But that also makes it mysterious why Dos Santos thinks we see it. As a rule, we perceive entities in the world – for instance, the apples on our table. We do not perceive our own sensations – our sensations of the apples, say. We *have* these sensations; but we do not *perceive* them. To suppose otherwise is to posit a much-ridiculed "Cartesian theatre," whereby human perception involves the projection of sensations onto a screen in our heads that we view like a movie (Dennett 1991).

The upshot is that while Dos Santos can be commended for attempting to explicate the concepts of numerosity and numerousness—something most researchers who use these terms never bother to do—the resulting explication doesn't support a viable alternative to the thesis that the number sense represents plain-old numbers.

3. Format

We've been discussing the *content* of ANS representations — *what* they represent. But researchers are also interested in the *format* of ANS representations — *how* they represent. Analog thermometers and digital thermometers both represent temperature; but they use distinct formats to do so. Knowing that the ANS represents numbers does not settle its format.

3.1 The Argument from Weber's Law

There is a powerful argument suggesting that the ANS employs an analog format. The argument begins from the observation that numerical discrimination obeys Weber's Law: as noted, ANS discrimination of two numbers is a linear function of their ratio. *Ceteris paribus*, it's easier to discriminate 7 dots from 8 dots than 8 dots from 9 dots even though the difference in both cases is just one dot. That's because the ratio 7:8 is further from 1:1 than the ratio 8:9.

Weber's Law can be readily explained if the ANS uses an internal magnitude, such as a neural firing rate, which increases or decreases as a monotonic function of the numerical values represented. For, in this case, similar numbers will be represented by physically similar magnitudes. So long as the system is subject to noise of an appropriate sort, subjects would then be more likely to confuse two numbers as their ratio approaches 1:1.

Suppose that 20 items were represented by a mean neural firing rate of 20 Hz, 21 items by a mean of 21 Hz, etc. Because of neural noise, a display of 20 dots would not always elicit a response of 20 Hz. Instead, the response might take the shape of a bell curve. Most often, 20 dots would elicit a response of 20 Hz; but it would sometimes elicit a response of 19 Hz or 21 Hz, less often 18 Hz or 22 Hz, etc. And the same will be true of the response curves for other numbers. Nearby numbers will thus have overlapping response curves and discrimination will deteriorate as the ratio of the two numbers approaches 1:1.

This quick sketch gives the general idea: if we assume an analog mapping between number and an internal magnitude, we get a natural explanation for why ANS-based discrimination follows a pattern. But to make the math work and get Weber's Law (such that discrimination is a linear function of ratio) the analog mapping and noise need to match in the right way. This is where things get complicated, since there are infinitely many mappings we might appeal to. Nevertheless, some possibilities seem more natural than others.

For example, a logarithmic mapping (wherein the internal magnitude increases logarithmically with number) can be paired with additive noise to yield Weber's Law. Additive noise is simple and there's a tradition, tracing back to Fechner, of describing psychophysical relations logarithmically. Another popular option is to pair a linear mapping with multiplicative noise. Linear relations are especially simple, and it isn't hard to imagine natural processes that would yield multiplicative noise. Alternatively, because neural noise is often believed to obey a Poisson distribution, some theorists have sought to explain Weber's Law with a pairing that marries Poisson noise to a third mapping — one that squares the sum of the logarithm of the stimulus and a constant (Zhou et al. 2024).

Because these three pairings are relatively natural, they have been considered strong candidates by which to explain Weber's Law. For now, the important point is that they all presuppose some kind of analog mapping between number content and an internal vehicle: as number increases or decreases, so does the internal magnitude. The ANS thus appears to have an analog format, a bit like a mercury thermometer.

Two points are worth emphasizing. The first is that 'analog' here does not mean continuous. We can explain Weber's Law whether we take the internal magnitude to be continuous like a firing rate, or discrete like the number of neurons firing above some threshold. All that matters is that the internal magnitude mirrors, or covaries with, number (Maley 2011). The term 'analog' is thus being used in the *mirroring sense* rather than the *continuous sense*, like when we say that a wall clock whose second hand advances in discrete (rather than continuous) steps is analog since the angle it traces mirrors the time elapsed (Beck 2018, 2019). The important idea is that the postulation of analog vehicles in the mirroring sense offers to explain Weber's Law in a way that the postulation of digital constituents, which bear arbitrary relations to their contents, does not.

Second, 'analog' in the present sense does not presuppose the "parts principle," the idea that if X represents Y, then the parts of X must represent the parts of Y (Kosslyn 1980; Fodor 2007; Carey 2009). Suppose, once again, that the internal magnitude is a neural firing rate. 20 Hz does not have 19 Hz as a part in any non-metaphorical sense. Thus, a representation can be analog in the mirroring sense without obeying the parts principle (Ball 2017; Peacocke 2019; Clarke 2022a; 2023).

3.2 Gallistel on Autoscaling

C.R. Gallistel was among the first to clearly articulate a version of the above argument that the ANS has an analog format (Gallistel & Gelman 1992, 2000; Gallistel, Gelman, & Cordes 2006). It is thus interesting to note that Gallistel (2011; 2017) has more recently defended a new view about the format of magnitude representations that he calls *autoscaling*.

Gallistel is animated by concerns with existing explanations of Weber's Law. As noted, existing explanations have two parts: a mapping of number content to an internal magnitude and a noise distribution—for example, a logarithmic mapping with additive noise or a linear mapping with multiplicative noise. Gallistel worries that there is no plausible physical implementation of the linear mapping because thinkers need to represent numbers over a vast dynamic range. "Probabilities, for example, must be represented by numbers much less than 1. On the other hand, cognitive maps of familiar terrain span distances from centimeters to thousands of kilometers" (2017, p. 6; see also Gallistel this volume p. x). Any internal magnitude that implemented a linear mapping over so many orders of magnitude would need implausibly many differentiable values.

A logarithmic mapping would be an improvement in this respect. Since a logarithmic scale compresses larger values, it can handle a large dynamic range. But Gallistel argues that it has other limitations. For example, while a logarithmic code easily handles multiplication and division, it struggles with additions or subtractions that the ANS is known to facilitate (Barth et al. 2006). Multiplication can be reduced to addition since the log of the product of two values is just the sum of their logs. But "[u]nless recourse is had to look-up tables, there is no way to implement addition and subtraction, because the addition and subtraction of logarithmic magnitudes corresponds to the multiplication and division of the quantities they refer to" (2011, p. 8). Gallistel further objects that logarithms cannot represent sign (negative as well as positive magnitudes) because negative numbers are reserved to represent values between zero and one. Finally, Gallistel worries that logarithms cannot represent 0 because the logarithm of 0 is negative infinity.

These problems lead Gallistel to defend his autoscaling account, according to which "every mental magnitude is bipartite, like the representation of quantity in scientific notation: one part specifies where the magnitude falls within some range; the other specifies the range (scale)." Thus, consider a representation of the number 325. We can write this in scientific notation as 3.25×10^2. Here, the range, or scale, is specified by the base and exponent, 10^2. This tells us that we're dealing with a number in the hundreds. The precision, or where the number falls within that range, is determined by the significand—in this case, 3.25. In this example, the significand is specified to two decimal places. Given the range, this means that it will be able to distinguish up to the ones place—e.g., 325 from 326. But it won't be able to distinguish beyond that—e.g., 325.1 from 325.2. Suppose we instead wanted to represent 32,500. Using scientific notation, and keeping the precision the same, we would write 3.25×10^4. This time the scale says that our number is in the ten-thousands. The significand again tells us where in that range our number falls, but this time it is valid only to the one-hundreds place. It can distinguish 32,500 from 32,600; but it cannot distinguish 32,510 from 32,520.

Crucially, the scale, as modeled by the exponent, is dynamic. It can be adjusted to the context. If you see a large number of dots, your ANS will use a larger scale. If you see a smaller number of dots, your ANS will use a smaller scale. Gallistel argues that this autoscaling approach can explain Weber's Law. For if the precision is constant (if the significand is always specified to the same number of decimal places), the imprecision will scale with the exponent. For example, 0.25×10^2 and 0.25×10^4 are each precise to within 1%—and the same would be true for any exponent given a two-place significand in base-ten notation. By contrast, if the significand were only specified to one decimal place, the values would only be precise to within 10%. Our ability to discriminate two numbers could thus be determined by their ratio, much as Weber's Law predicts.

3.3 Logarithmic and Linear Models Reconsidered

Gallistel's defense of autoscaling has two parts: a critique of existing (log and linear) coding schemes; and a positive vision of what should replace them. We now evaluate each part in turn.

Consider Gallistel's critique of logarithmic coding schemes first. Gallistel suggests that because adding logarithms is difficult, the brain wouldn't use a logarithmic coding scheme. But we shouldn't assume that the brain will implement computations over functions that we find easy or natural. While *we* need a look-up table to add logarithms, a neural network might well implement addition over logarithms without too much trouble. Gallistel also argues that logarithmic coding schemes can't handle sign, but we don't see why the brain couldn't use opponency channels for positive and negative magnitudes. Finally, while a strict logarithmic mapping cannot represent zero, the brain might instead use a serviceable approximation. What's important is that the representation of number is compressed in a roughly logarithmic way, not that it literally follows the familiar mathematical function of a logarithm by approaching negative infinity at zero.

While we aren't persuaded by Gallistel's armchair arguments against a logarithmic mapping, there is empirical evidence we find suggestive. If number were represented logarithmically, multiplication over ANS representations should be easy since log(m x n) = log(m) +log(n). But while ANS-based multiplication has been observed (Qu et al. 2021), a study by Pickering et al. (2022) tested adults' abilities to multiply ANS representations by numbers between 2 and 8 and found that while they were above chance in multiplying approximately enumerated quantities by 2-4, such that the resulting estimations erred in ways that conformed to Weber's Law, performance fell to chance with numbers 5-8. Since a capacity of four items is a signature of working memory and subitizing, this suggests that multiplication requires working memory. An ANS representation can be multiplied through a subitizable number of successive addition operations, but "two ANS representations cannot be multiplied together" (Pickering et al. 2022, 1). While we don't take this to close the door on a logarithmic mapping—alternative explanations remain possible—we consider it suggestive that a linear (or non-logarithmic) mapping is used in certain contexts (see below).

Next, consider Gallistel's objection that a linear mapping cannot handle the necessary dynamic range of magnitude content. While Gallistel is surely right that distances and durations are represented over many orders of magnitude, the ANS may be more limited. When deployed visually, for example, items need to be sufficiently spaced. When they aren't, discrimination is still possible, but it no longer exhibits Weber's Law, the psychological signature of the ANS. Instead, it exhibits a square root law, suggesting that a separate texture density system is utilized (Anobile et al. 2014). Given these constraints, the ANS might have an upper limit of 200 or 300 items (ibid.). Conversely, while it's true that the ANS can represent rational numbers smaller than 1 (Clarke & Beck 2021a), existing evidence suggests that these are represented using ANS representations of whole (positive) integers, composed into a part/whole format like a fraction (Qu et al. 2024). The linear model thus seems like a live option for the format of ANS representations.

At this point it's worth flagging a background assumption, which often goes unnoticed, but pervades work on the psychophysics of magnitudes. To appreciate this assumption, notice that, in much of his work, Gallistel emphasizes that representations of different magnitude types need to play nice with one another. For example, Gallistel has regularly argued that representations of time and representations of distance need to be multipliable in creatures which estimate velocity; likewise, the representation of number and time need to be capable of being divisible by one another to compute rate. That might, in turn, seem to require that these representations all have the same format, and one which is well-suited to multiplication and division. Thus, if distance (or duration) representations cover too many orders of magnitude for a simple linear mapping, and if linear mappings handle multiplication and division poorly, we shouldn't expect ANS representations to be linearly mapped either. But this line of reasoning presupposes a hidden assumption – *the single-*

format assumption – which holds that all magnitude representations share a single format that is employed in all contexts of use. We think this assumption is dubious.

We've already seen that ANS representations must play myriad roles. In some contexts, they facilitate simple addition and subtraction operations. By Gallistel's lights, such operations are well-served by analog representations with a linear mapping, and not by analog representations with a logarithmic mapping. But we have now seen that, in other contexts, the ANS interacts with representations of other magnitude types and enters into division or multiplication operations with these. Arguably, these functions are better served by analog representations with a logarithmic mapping. The upshot is that analog representations with different mapping functions between content and vehicle are needed if these are to be well-suited to their varied task domains.

This might seem puzzling were it not for the fact that it is common practice in many scientific fields, including machine learning and statistics, to regularly translate into and out of logarithmic representations.[4] One reason is that translating into logarithmic representations can massively simplify the math. For example, multiplication can be reduced to addition and exponents can be eliminated. Another reason concerns numerical stability. When calculating probabilities, the numbers can quickly become tiny because repeatedly multiplying small probabilities brings one closer and closer to zero. Precision thus becomes a worry: a number that is too close to zero can be mistaken for zero, disrupting the algorithm. Translating to logarithms avoids this problem since it turns the tiny numbers into ordinary negative numbers that can then be added together. Of course, once the algorithm runs its course, the output can be translated back from a log to a value between 0 and 1. But here's the point: if researchers routinely translate into and out of logarithmic representations for these and other reasons, we should be open to the possibility that the brain might do something similar.

And indeed, we already know the brain translates across formats. The visual system flexibly translates between polar and cartesian coordinates when representing small scale spatial relationships (Yousif & Keil 2021). Similarly, spatial memory translates between egocentric and allocentric representations (Burgess 2006; Wang et al. 2020). Indeed, evidence already exists that the ANS makes use of multiple formats. Siegler & Opfer (2003) found that children sometimes map number to space logarithmically and sometimes linearly. Similarly, Dotan & Dehaene (2013) had numerate adults point to the position of a number on a number line. Looking only at the final position indicated, the pointing was best modeled linearly. But the pointing motion was also found to contain an initial, transient trajectory that was best modeled logarithmically, suggesting that the subjects began their act of pointing using a logarithmic code but then quickly converted to a linear code that guided the act's completion. More recently, Ratcliff & McKoon (2018) compared two tasks. In the first, subjects had to decide which of two side-by-side displays of dots, all the same color, was greater in number. In the second, the dots came in two colors and were intermingled in a single display, with the task being to say which color corresponded to the greater number of dots. They found that a log model better fit reaction time and accuracy data for the side-by-side task but that a linear model was a better fit for the intermingled task, suggesting that the ANS exploits different formats in different contexts.[5] We have good reason to question the single-format assumption.

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[4] We thank Daniel Herrera for discussion of the points to follow in this paragraph.

[5] Darko Odic (pers. comm.) has unpublished data that conflict with Ratcliffe and McKoon's results. But our point isn't that their study definitively shows that multiple formats are employed by the ANS; it's simply that we should be on the lookout for evidence that tests this hypothesis.

3.4 Evaluating Autoscaling

Even if linear and logarithmic coding schemes, or some combination of the two, remain viable formats of ANS representation, Gallistel's account of autoscaling deserves consideration on its own merits. As we've seen, this account is grounded in an analogy with scientific notation. We want to distinguish two interpretations of the analogy, one modest and one strong.

On the modest interpretation, the analogy is simply meant to suggest that the ANS uses two components: an *interval* of fixed precision (analogous to the significand) and a contextually determined *scale* (analogous to the base and exponent). The interval has a set number of values; but what those values mean is determined in context by the scale. By analogy, consider a line with ten equally spaced markings. Each marking represents a different magnitude value. But which values are represented depends on the context. Similarly, your ANS might use a single interval—say, firing rates that vary from 11 Hz to 20 Hz—with what those rates signify varying with context. In some contexts, the interval of 11 Hz to 20 Hz might correspond to the numbers 1 to 10; in other contexts, it might correspond to 1 to 100; and so on. On this interpretation, autoscaling is a modified version of the linear mapping account that overcomes the implementation worry through dynamic scaling.

That's the modest interpretation. The strong interpretation is that the ANS uses representations that are like scientific notation not only in being bipartite, but in being fully digital. On this view, there is no need to posit analog representations. Weber's Law can be fully explained by digital representations that are analogous to scientific notation in their bipartite structure. Although we are unclear which interpretation Gallistel intends to endorse in his rich discussion of autoscaling, a virtue of the strong interpretation might be that it offers to explain the myriad functions of the ANS by appeal to a single format type, since digital representations ably facilitate such varied functions as addition, subtraction, multiplication, and division. By contrast, we saw that the straightforward implementation of these functions on the modest interpretation might require analog representations with a plurality of mapping functions and formats. Without independent motivation, this could sound messy and unnecessarily cumbersome.

Despite the simplicity of the strong interpretation we believe the modest interpretation is more plausible. First, the evidence advanced in support of autoscaling only supports the modest view. For example, Gallistel cites a study showing that a neuron in the fly's visual system tuned to angular velocity rapidly adjusts its activity with context. As conditions become more or less turbulent, the neuron's activity normalizes to these conditions in a nearly optimal way, while also encoding the scale. This is evidence of a bipartite representation as the modest interpretation maintains; but it is silent on whether the representations are digital.

Another study—the only one we know of to explicitly test Gallistel's autoscaling hypothesis—also appears to be directed at the modest interpretation. Odic et al. (2024) presented subjects with several randomly oriented lines of different lengths, each of a different color (Figure 2). The lines were flashed on a computer screen for two seconds and subjects were then asked to report the color of an ordinally identified line—for example: the longest line; the second longest line; the middle-length line; the shortest line; etc. The researchers reasoned that if length was encoded in a bipartite format, then the scale would be set by the contextually determined endpoints—the maximum and minimum values. The shortest and longest lengths should thus be privileged. Subjects should be faster, and make fewer errors, identifying them. Furthermore, performance identifying the shortest and longest lines might be unaffected by group size (the number of lines) since they are identified in virtue of setting the scale. So, whether there were five lines, seven lines, nine lines, or eleven lines, subjects should be just as accurate and fast at identifying the longest and shortest lines. By contrast, if the ordinal position of each line were determined by a series of pairwise comparisons, it should be harder and take longer to identify the longest and shortest lines as group size increases. Both predictions were confirmed. Moreover, the findings persisted in follow-up studies when the researchers substituted area or number for length. Subjects were always fastest and most accurate at identifying the endpoints; and their ability to do so was unaffected by group size. And although one might initially worry that performance was a byproduct of how the task was phrased (the longest and shortest lines were explicitly named), the findings held up when subjects were asked to find the third line above/below the middle, instead of the longest/shortest.

Figure 2: Sample stimuli from Odic et al. (2024). From left to right: Five lines of variable length; Seven lines of variable length; Dots of variable size; Collections of variable number. In each case, people find it easiest to identify the largest and smallest items/collections in each array.

These findings are suggestive of a bipartite representation since the scale used, as determined by the endpoints, appears to be processed separately from the interval. But the findings don't suggest that the representations are digital. In fact, there's some reason to think them analog. While the findings show that the endpoints are privileged, Odic et al. do not explain *how* the endpoints are identified in the first place if not through a serial comparison with each and every value. But analog computations are an obvious solution. If you want to know which piece of spaghetti is the longest, you don't need to serially compare them. You can just bang them on the table and see which one sticks out furthest (Clarke 2022b). That's a paradigmatic analog computation, which turns on the fact that spaghetti lengths mirror (in this case, are identical to) the magnitudes we're interested in probing. And the fact that subjects in Odic et al.'s experiment get the endpoints for free suggests that they too might be making use of analog computations over analog representations.

To date, evidence in favor of autoscaling thus appears to support the modest interpretation at most. But there's a second reason to doubt the strong interpretation of Gallistel's hypothesis: its explanation of Weber's Law is incomplete. On the strong interpretation, effects of Weber's Law emerge each time the scale is changed. By analogy with scientific notation, increasing the exponent decreases the precision by a fixed percentage. But given a fixed scale, the precision should be fixed. For example, given base 10, there should be no manifestations of Weber's Law for quantities 0-9. But Weber's Law shows up for any two values, even once the range is fixed by the context. For example, Odic et al. found that even once the endpoints were selected, setting the range, performance decreased linearly with the ratio of the lengths (or areas or numbers), just as Weber's Law predicts. We conclude that while the modest autoscaling account is viable and plausible, there remains reason to think that ANS representations of number are couched in an analog format.

4. Conclusion

The present chapter has sought to clarify and extend our position on the content and format of ANS representations. Building on our suggestion that the ANS represents *numbers*, we have considered how this relates to proposals that the system simply represents cardinalities, numerosities, or numerousness, explaining why we continue to think these deflationary proposals fail to offer clear or viable alternatives to our account. We have also extended prior discussions of ANS format, introducing the suggestion that the system employs and flexibly transitions between analog representations couched in diverging formats (e.g., employing linear and logarithmic mappings respectively). Finally, we have suggested that while Gallistel's recent postulation of a bipartite ANS format is tenable, plausible incarnations of this hypothesis still require numerical representations with an analog format if they are to predict or explain the ANS's conformity to Weber's Law.

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