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The number sense represents (rational) numbers

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Abstract

On a now orthodox view, humans and many other animals possess a "number sense," or approximate number system (ANS), that represents number. Recently, this orthodox view has been subject to numerous critiques that question whether the ANS genuinely represents number. We distinguish three lines of critique – the arguments from *congruency*, *confounds*, and *imprecision* – and show that none succeed. We then provide positive reasons to think that the ANS genuinely represents numbers, and not just non-numerical confounds or exotic substitutes for number, such as "numerosities" or "quanticals," as critics propose. In so doing, we raise a neglected question: numbers of what kind? Proponents of the orthodox view have been remarkably coy on this issue. But this is unsatisfactory since the predictions of the orthodox view, including the situations in which the ANS is expected to succeed or fail, turn on the kind (s) of number being represented. In response, we propose that the ANS represents not only natural numbers (e.g., 7), but also non-natural rational numbers (e.g., 3.5). It does not represent irrational numbers (e.g., $\sqrt{2}$), however, and thereby fails to represent the real numbers more generally. This distances our proposal from existing conjectures, refines our understanding of the ANS, and paves the way for future research.

1. Introduction

On a now orthodox view, humans and many nonhuman animals possess a *number sense*, or *approximate number system* (ANS), that affords a primitive and prelinguistic capacity to represent number. This hypothesis is supported by a large, and ever-growing, body of experimental research. But, at its core, the orthodox view is dogged by two significant challenges.

First, many deny that the ANS actually represents number. Instead, critics claim that the ANS merely represents non-numerical confounds (Gebuis, Cohen Kadosh, & Gevers, 2016; Leibovich, Katzin, Harel, & Henik, 2017) or exotic substitutes for number, such as "numerosities," "quanticals" (Núñez, 2017), or "pure magnitudes" (Burge, 2010). Consequently, a cloud of uncertainty hangs over the orthodox view. Perhaps, there is no *number* sense after all?

Second, even if the ANS does represent number, a question remains: What *kind(s)* of number does it represent? Proponents of the orthodox view have been remarkably coy on this issue. The default hypothesis seems to be that the ANS represents natural numbers – that is, positive whole numbers (e.g., 7) – but this is rarely made explicit. An alternative is that the ANS represents real numbers, which include not only natural numbers, but also rational numbers (e.g., ½) and irrational numbers (e.g., $\sqrt{2}$) (Gallistel & Gelman, 2000). There are other options as well, such as the hypothesis that the ANS represents rational but not irrational numbers. Choosing among these is important since the predictions of the orthodox view, including the situations in which the ANS is expected to succeed or fail, turn on the kind(s) of number being represented. Thus, without a clear sense of what the ANS represents, the orthodox view remains vague and underspecified.

The present treatment addresses both challenges on behalf of the orthodox view. After introducing the ANS in more detail and issuing some clarifications (sect. 2), we distinguish and address three lines of critique that have motivated skepticism about the ANS and its capacity to represent number – critiques which we label the arguments from *congruency*, *confounds*, and *imprecision* (sects 3–5). We then highlight positive reasons for thinking that the ANS literally represents numbers, of a sort familiar from the math class, rather than ersatz numbers (e.g., "numerosities") (sect. 6). In so doing, we raise the neglected question: numbers of what kind? In answer to this question (sect. 7), we marshal evidence that the ANS represents both natural numbers and non-natural rational numbers. At the same time, we argue that the ANS fails to represent irrational numbers, and thus fails to represent the real numbers more generally. This distances our proposal from existing conjectures, refines our understanding of the ANS, and paves the way for future research on this topic.

2. The ANS

The ANS is a primitive and prelinguistic numerical system that is marked by a distinctive performance profile. For while the ANS facilitates computations over (sometimes quite large) numerical quantities, its numerical discriminations are imprecise and conform to Weber's Law. As such, discrimination deteriorates as the ratio between two numbers approaches 1:1. Thus, 8 is easier to discriminate from 10 than 10 is from 12 even though the absolute difference in number is the same. Meanwhile, 10 is as easy to discriminate from 5 as from 20. In each case, discriminability is determined by the ratio of the two quantities.

This performance profile distinguishes the ANS from other psychological systems that facilitate primitive numerical computations. For example, it distinguishes the ANS from a "parallel individuation" or "small number" system which facilitates *precise* numerical discriminations, but only among sets involving four items or less (Feigenson, Dehaene, & Spelke, 2004; Margolis, 2020; but see Cheyette & Piantadosi, 2020). For, unlike the parallel individuation system, the ANS's numerical discriminations are imprecise (they are ratio sensitive) and they are not limited to small sets of items (the system might recognize that 30 dots are more than 20 dots). Furthermore, it distinguishes the ANS from a "texture density system." This is a system which, unlike the ANS, enables organisms to discriminate sets that are too crowded to parse, and (thus) perceived as texture (Burr, Anobile, Togoli, Domenici, & Arrighi, 2019). Crucially, the texture density system's performance is not predicted by Weber's Law but by a psychophysically distinct square root law, revealing a marked difference in its signature limitations (Anobile, Cicchini, & Burr, 2016; Cicchini, Anobile, & Burr, 2016; Zimmermann, 2018).

2.1 Empirical motivations

The postulation of an ANS is not uncontroversial. Various critics deny that the system exists or that it genuinely represents number. Instead, they maintain that the relevant mechanisms and processes simply track non-numerical magnitudes, such as areas and densities, or recherché alternatives to number, such as "numerosities," as opposed to numbers themselves. In sections 3–5, we examine such critiques in detail. But, before we do, it is useful to consider some of the evidence that has (rightly or wrongly) motivated positing an ANS in the first place. Our focus will be on landmark studies and experimental paradigms that put naive readers in a position to appreciate where critics of the ANS are coming from, and where their critiques go wrong. For more comprehensive recent surveys, see Anobile et al. (2016), Anobile, Arrighi, Castaldi, and Burr (2021); Gebuis et al., 2016, Nieder (2016, 2020), and Odic and Starr (2018).

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2.1.1 Infant studies

In the early 2000s, various studies yielded evidence that young human infants can reliably track the numerical properties of large sets, albeit imprecisely and in accord with Weber's Law. For example, Xu and Spelke (2000) habituated 6-month-old infants to visual arrays containing either 16 or 8 dots. When habituated to an 8-dot array, infants recovered interest when presented with a 16- or 4-dot array, but not a 12-dot array. Meanwhile, infants habituated to a 16-dot array dishabituated to a 32- or 8-dot array, but not a 24-dot array. Because confounding variables such as brightness, density, and dot size were controlled for (Fig. 1a), these findings were interpreted as showing that 6-month-old infants can reliably discriminate the approximate number of items in two sets provided they differ by a suitably large ratio (e.g., 1:2). Subsequent studies suggested that these discriminative capacities improve with development. For instance, 9-month-olds were found to reliably discriminate sets that differ by a ratio of just 2:3 (Lipton & Spelke, 2003). In each case, performance decreased as the numerical ratio approached 1:1, irrespective of the sets' precise cardinal values.

2.1.2 Cross-modal infant studies

Cross-modal studies bolster the suggestion that these results reflect a genuine sensitivity to number. In one study, Izard, Sann, Spelke, and Streri (2009) found neonates capable of matching numerical quantities across sets even when they were presented with a number of *seen* items and a number of *heard* sounds (Fig. 1b). This complicates attempts to explain these findings in terms of a mere sensitivity to non-numerical confounds. After all, neonates in Izard et al.'s study could not have relied on (say) the size of seen items, or the total area of a seen set, when identifying a match, because properties of this sort could not have been heard. Nor could they have relied on the total volume or duration of heard stimuli, because properties of this sort could not have been seen. As such, these findings provide evidence that infants are able to abstract away from low-level confounds to track the numerical properties of sets they observe. And because controls indicated that these abilities are (again) ratio sensitive, they further implicate an ANS.

2.1.3 Preschooler studies

Of course, infants are notoriously difficult to study, requiring the use of indirect measures such as looking time. But preschoolers can simply be asked which of two stimuli has "more" dots or tones, and because they are too young to reliably count, they appear to rely on an ANS when answering. In a striking illustration of this, Barth, La Mont, Lipton, and Spelke (2005) showed that preschoolers could not only reliably answer which of two visual stimuli had "more" (e.g., red dots vs. blue dots), but that they were roughly *as good* at doing this across modalities (e.g., dots vs. tones) as within a single modality. This suggests that preschoolers' numerical competences are not tied to modality-

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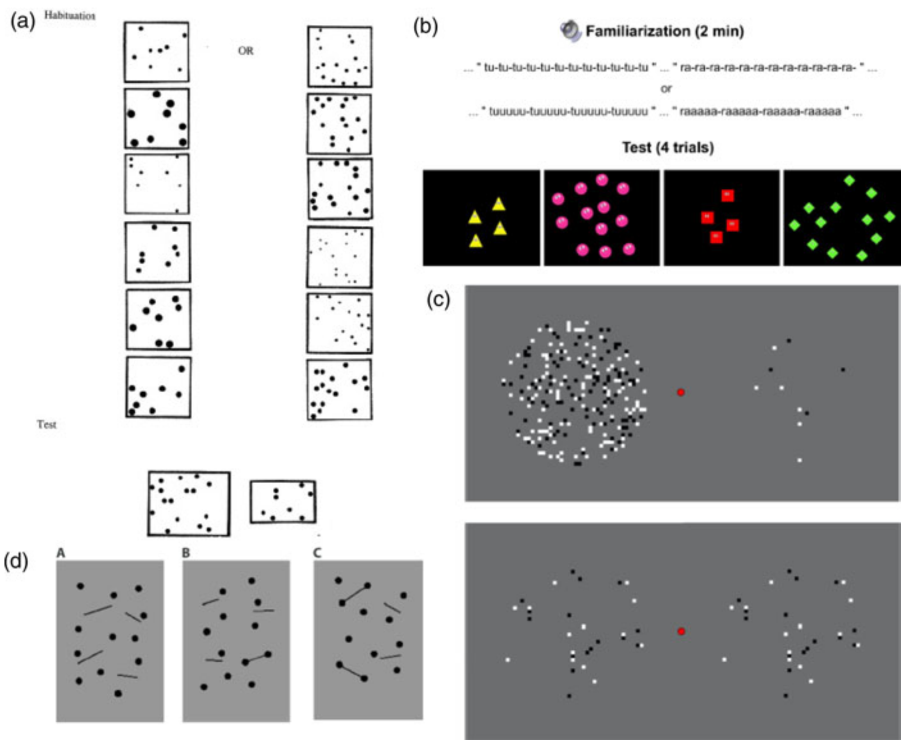

Figure 1. (a) Xu and Spelke habituated infants to arrays of dots and found that dishabituation would occur when test stimuli contained a number of dots that differed by a sufficiently large ratio. Because several confounds were equated in habituation displays (e.g., dot size) or test displays (e.g., density), this suggests a sensitivity to number itself. Reprinted from Xu and Spelke (2000, p. B5), Figure 1, Copyright © 2000, with permission from Elsevier. (b) Izard et al. familiarized neonates to a number of heard sounds before presenting them with visual arrays containing a number of items that either matched or failed to match the number of heard sounds. Neonates looked longer at visual arrays containing a number of items that failed to match the number of sounds they had been familiarized to. Reprinted from Izard et al. (2009, p. 10383), Figure 1, in line with PNAS's licensing agreement. (c) Burr and Ross instructed subjects to fixate on the dot in the top panel for 30 s, at which point the bottom panel was presented. Subjects reported that the display on the left appeared less numerous than the display on the right. Reprinted from Burr and Ross (2008), Supplementary Data, Copyright © 2008, with permission from Cell Press. (d) He et al. showed that connecting dots with thin lines substantially decreased the number of dots arrays were estimated to contain. Reprinted from He et al. (2009, p. 510), Figure 1, Copyright © 2009, with permission from Springer Publishing Company.

specific confounds. Once again, the numerical ratio between sets predicted performance, implicating an operational ANS in these children.

2.1.4 Adult studies

Studying adults adds a wrinkle to these investigations because most adults can use language to count, bypassing the ANS and its distinctive signature limitations. But when Barth, Kanwisher, and Spelke (2003) presented numerical stimuli too quickly for them to be explicitly counted, adults behaved like children. They discriminated stimuli in accord with Weber's Law under both intra- and inter-modal conditions. Similar results obtained during verbal shadowing tasks, where adults attempt to press a button a pre-specified number of times while repeating a word, such as "the," to prevent them explicitly counting button presses. In studies of this sort, errors in the total number of button presses increase in proportion to the pre-specified number of target presses and are, therefore, predicted by Weber's Law (Cordes, Gelman, Gallistel, & Whalen, 2001). So, again, these studies are indicative of a system with the performance profile of an ANS.

2.1.5 Adaptation studies

Burr and Ross (2008) adapted adult observers to a large or small number of dots by having them stare at a display for 30 s (Fig. 1c). They then presented them with new displays of dots. Observers who had adapted to a large number of dots underestimated the number of dots on the new display, while observers who had adapted to a small number of dots overestimated the number of dots on the new display. Observers also reported consciously experiencing the aftereffect: subsequent displays *appeared* more or less numerous. Although some have argued that these effects involve adaptation to density rather than number (Dakin, Tibber, Greenwood, Kingdom, & Morgan, 2011; Durgin, 2008), recent studies control for density and other confounds (DeSimone, Kim, & Murray, 2020; Fornaciai, Cicchini, & Burr, 2016). Indeed, some research suggests that numerical adaptation effects occur between auditory and visual stimuli and are thus cross-modal (Arrighi, Togoli, & Burr, 2014).

2.1.6 Dumbbell studies

In a different paradigm, subjects were presented with two visual arrays of items (squares or circles in Franconeri, Bemis, & Alvarez, 2009; dots in He, Zhang, Zhou, & Chen, 2009) and thin lines. Their task was to say which array had more items, ignoring the lines. When the lines connected two items (effectively turning pairs of items into single dumbbell-shaped objects), subjects underestimated how many items were in the array (see Fig. 1d). Franconeri et al. also showed that introducing a small break in the lines would substantially decrease this "dumbbell effect." Because displays with small breaks and displays of items connected with thin lines differ only slightly with respect to total surface area, spatial frequency, and other non-numerical magnitudes, these studies are (again) indicative of a genuine sensitivity to *the number* of objects observed.

2.1.7 Number neurons

Finally, all of these results can be linked to findings at the level of neural implementation. Neurons in the intraparietal sulcus (IPS) of monkeys respond selectively to specific numbers (Nieder, 2016). Thus, specific neurons in the IPS respond preferentially when one perceives *seven* items. Interestingly, their response profile is noisy. Thus, neurons which are tuned to seven often fire when one observes six or eight items and occasionally when one observes five or nine items. Indeed, noise levels *increase* with numerical quantity. This is what we might expect of neurons implementing an ANS because it seems to naturally explain why conformity to Weber's Law emerges at the behavioral level.

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2.2 Clarifications

The preceding remarks do not provide a comprehensive overview of research on the ANS. Nor do they prove its existence. They simply provide readers with an initial sense of the vast, and seemingly convergent, evidence that has motivated positing an ANS in humans that represents number. The rest of this paper will be devoted to defending and extending this conjecture. But, before we proceed, it's important to clarify our aims.

2.2.1 Auxiliary claims

There are certain hypotheses that frequently accompany the postulation of an ANS which we can set aside for current purposes. These include hypotheses that the ANS is innate, that it is phylogenetically widespread, and that it is homologous across certain species. We can also remain largely neutral on the format of ANS representations (Beck, 2015, 2018; Clarke, forthcoming), the informational resources it accesses in its computations (Mandelbaum, 2013; Margolis & Laurence, 2008), and the details of its neural implementation (Lucero et al., 2020; Nieder, 2016). Our question is simply whether the ANS represents numbers, and if so, numbers of what kind?

2.2.2 Referents versus modes of presentation

We should distinguish *the referents* of ANS representations (*what* they represent) from their modes of presentation (*how* they represent). By analogy, the gustatory system plausibly represents sodium chloride (NaCl) concentrations. But it is a further question whether it represents NaCl concentrations *as such* – that is, as comprising molecules constituted by sodium and chloride atoms. Perhaps it is better to say that the gustatory system represents NaCl under the mode of presentation *salty*. We will argue that the ANS represents numbers (i.e., that numbers serve as the referents of the ANS), but under a unique mode of presentation that respects the imprecision inherent in the ANS (sect. 6). This will allow us to avoid a commitment to exotic entities such as "numerosities" without losing sight of the important differences between ANS representations and the precise numerical concepts that emerge later in development.

2.2.3 Perceiving numbers

Some may wonder how the ANS could refer to numbers given that numbers are abstract objects, not located in space or time. This worry might seem especially acute given that the ANS often operates perceptually (as in the adaptation studies introduced above). For how could anyone *perceive* the number seven? To clarify, note that philosophers standardly take perception to have an *object–property structure*. Perceptual states refer to objects, which are concrete particulars, and attribute properties to them. To perceive an object as red and square is to attribute the (abstract) properties of redness and squareness to a spatiotemporally located object. Among other virtues, this allows us to say what different perceptual states have in common. For example, the perception of a red apple and the perception of a red barn door concern different particulars but are alike insofar as both attribute redness. On this standard picture, abstract objects enter perceptual content through the attribution of properties, not through reference to objects. Likewise, if the ANS refers to numbers, it does so by enabling numbers to enter into contents via property attribution, not as objects of perception. Therefore, your ANS might enable you to perceive the collection of apples on the table as being seven in number, but it wouldn't enable you to perceive the number seven itself – on its own, as it were – as an object.

2.2.4 Direct versus indirect models of the ANS

Consider two distinct approaches to modeling the ANS. On a *direct* approach, the ANS fulfills its numerical function by individuating entities in the world and then counting these up. For instance, Dehaene and Changeux (1993) propose that the ANS performs an initial process of "normalization," which identifies individual items independently of confounding variables, such as size and density. A later "accumulator" stage of processing then sums these, such that number is estimated. By contrast, an *indirect* approach eschews the initial individuation of items in favor of heuristic processes that derive number from continuous magnitudes. Thus, Dakin et al. (2011) and Morgan, Raphael, Tibber, and Dakin (2014) propose that numerical quantity is estimated in visual perception on the basis of spatial frequency information at high and low bandwidths and is recovered in one step from information about area and density.

Although we believe that various phenomena, such as the dumbbell effects and cross-modal comparisons discussed in section 2.1, implicate a direct model of ANS processing (Anobile et al., 2016), at least in part, we do not consider this issue settled. It is, thus, important to stress that our defense of an ANS with genuine number content does not presuppose a direct model. This bears emphasizing because critics sometimes assume that if indirect models of the ANS are correct, then the ANS could represent nothing more than the continuous magnitudes (e.g., areas and densities) on which its computations would be based. This is a mistake. Perceived depth is computed on the basis of cues such as binocular disparity, motion parallax, and retinal accommodation. But, of course, the visual system represents depth, and not (just) these cues. Correspondingly, the ANS might function to represent number even if its numerical estimations are based on a diverse range of continuous magnitudes and not on a representation of the individuals it functions to enumerate (Halberda, 2019). To suppose otherwise is to confuse *what* the system is doing (e.g., functioning to track and represent numbers – the *computational level* description with which we are concerned) for a specific account of *how* it does this (an *algorithmic level* description; Marr, 1982).

With these clarifications in view, we will now address three critiques which have targeted the ANS's alleged capacity to represent number (sects 3–5), offer positive reasons to think the system does represent number (sect. 6), and consider the specific kinds of number that it represents (sect. 7).

3. The argument from congruency

One reason critics doubt the existence of an ANS, with numerical content, derives from *numerical congruency effects*, where numerical judgments are influenced by the perception of irrelevant magnitude types. For instance, when subjects compare Arabic numerals and decide which picks out a larger number, their reaction times are influenced by font size. Therefore, when the larger numeral is printed in a larger font (a "congruent" trial), they answer more quickly than when numerals are identical in size (a "neutral" trial). And when the smaller numeral has a larger font (an "incongruent" trial), they are slower and less accurate (Gebuis, Herfs, Kenemans, de Haan, & van der Smagt, 2009; Gebuis, Kenemans, de Haan, & van der Smagt, 2010; Henik & Tzelgov, 1982). Similar effects occur in non-symbolic tasks

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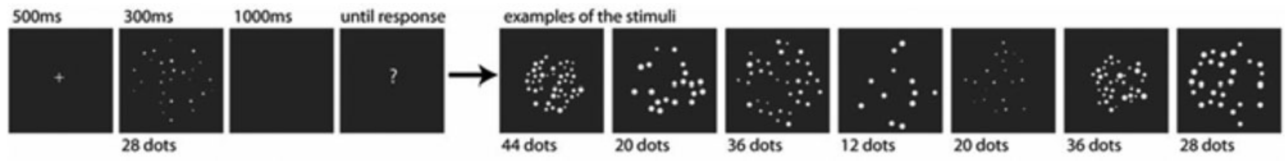

Figure 2. After fixation, subjects saw a display of dots briefly flashed on a screen. When the question mark appeared, they had to estimate the number of dots flashed by typing a number on their keyboard. The displays to the right of the arrow show examples of the stimuli that were used in this estimation task. Gebuis and Reynvoet found that numerical estimations were strongly influenced by non-numerical confounds, such as convex hull (a large convex hull disposed subjects to overestimate the total number of dots). Reprinted from Gebuis and Reynvoet (2012c, p. e37426), Figure 1, in line with PLoS ONE's adopted CC BY License.

(Fig. 2). Thus, subjects tasked with determining whether one dot display is more or less numerous than another are influenced in comparable ways by things such as average dot diameter, density, convex hull (the smallest convex area enclosing all the dots), and brightness (Cohen Kadosh & Henik, 2006; Dakin et al., 2011; Gebuis & Reynvoet, 2012a, 2012b; Leibovich & Henik, 2014).

The *argument from congruency* proposes that these effects undermine the existence of an ANS that genuinely represents number (Gebuis et al., 2016; Leibovich et al., 2017). For if there were an ANS which represents number, we would expect relevant numerical judgments to be based *entirely* on its outputs. Thus, irrelevant non-numerical magnitudes should be ignored. But the existence of congruency effects suggests that (often) they're not. Or as Gebuis et al. (2016, p. 22) put it: If relevant numerical judgments are influenced by the perception of non-numerical magnitudes, then

> why would there be an ANS system that can extract "pure numerosity"? What would be the use of having a system that can tell us exactly which cue [*sic*] at the passport control contains less people when it in the end adjusts this accurate answer in a possibly incorrect answer [*sic*] when for instance the length of the people in the cue [*sic*] is taken into account?

From the perspective of optimal design, Gebuis et al. propose that it makes little sense for a dedicated ANS to exist if its outputs are influenced by confounding variables in this way.

This objection presupposes that the ANS isn't governed by an indirect model that makes use of non-numerical cues such as convex hull and length. If it were, the ANS would obviously be influenced by those cues. But perhaps this is a safe assumption. Standard indirect models do not appeal to these cues, and nor, of course, do direct models.

Still, the argument faces other difficulties. An initial problem is that it overgeneralizes. It's well known that congruency effects affect judgments of uncontroversially perceptible magnitudes. For instance, judgments of duration exhibit congruency effects on size (Lourenco & Longo, 2010; Xuan, Zhang, He, & Chen, 2007), luminance (Xuan et al., 2007), length (Casasanto & Boroditsky, 2008), and distance (Sarrazin, Giraudo, Pailhous, & Bootsma, 2004). Therefore, if congruency effects demonstrate that numerical quantities are not represented by the ANS, then by parity of reasoning they would demonstrate that paradigmatically perceptible magnitudes (such as duration and distance) are not perceptually represented either.

To compound matters, congruency effects tend to be symmetric. For while numerical judgments are influenced by area and density, judgments of area and density are similarly influenced by number. Indeed, number often influences judgments of area and density *at least as much* as vice-versa (Cicchini et al., 2016; Savelkouls & Cordes, 2020; but see Yousif & Keil, 2020). So, if the fact that numerical judgments are influenced by area and density shows that number is not represented by an ANS, there should be equal or greater evidence that area and density are not represented either. In this way, the argument from congruency leads to an implausible skepticism about the perceptual representation of magnitudes quite generally.

These considerations indicate that the argument from congruency fails, but where does it go wrong? The argument errs in assuming that congruency effects are even in tension with the existence of an ANS, as we understand it. If there is an ANS which serves to represent numerical quantities, the observation of continuous magnitudes might introduce biases at the initial encoding stage, influencing the inputs the ANS receives, or at decision/response stages, altering outputs of the system. For example, at the response stage, congruency effects might reflect a Stroop-like byproduct of competition for a single response. In classic versions of the Stroop task, color and semantic processing compete for a single behavioral response (Johnson, 2004). Numerical and non-numerical magnitudes might compete for a behavioral response in similar ways (see Picon, Dramkin, & Odic, 2019).

Alternatively, at the decision stage, congruency effects might reflect a useful strategy. For given Weber's Law, the ANS is not perfectly precise. To counteract that imprecision, subjects might make use of correlations among numerical and non-numerical magnitudes, perhaps in a Bayes' optimal way (Petzschner, Glasauer, & Stephan, 2015; see also Martin, Wiener, & van Wassenhove, 2017). As Content, Velde, and Adriano (2017, p. 20) note, "Continuous dimensions are indeed most often correlated with number in our experience of the world. No wonder that we would tend to use them, when possible, in comparing collections."

Gebuis et al. (2016, p. 22) object that while "the majority of studies show that larger sensory cues cause an overestimation [in number] and smaller sensory cues an underestimation, exceptions exist." For instance, Gebuis and Reynvoet (2012a) found that total surface area was sometimes inversely correlated with numerical estimation. This is something Gebuis et al. (2016) consider an embarrassment for proponents of the ANS who try to explain away congruency effects in terms of biases, because biases should (allegedly) be the same on all occasions. But we are not sure how puzzling this really is for proponents of the ANS. A Bayes' optimal inference might be expected to treat different magnitudes differently, and to even treat the same magnitudes differently in different contexts (the same premise does not always license the same conclusion!). But, even if this were not so, such considerations only appear *more* puzzling for skeptics of the ANS. The fact remains that numerical judgments are generally

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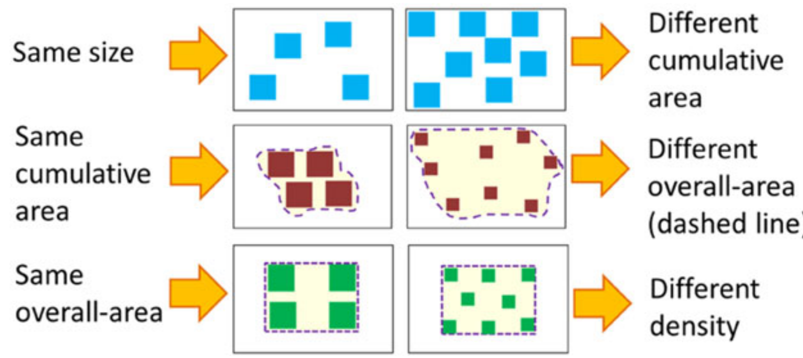

Figure 3. Attempts to control for one non-numerical confound often leave other confounds uncontrolled for. For example (top row), if two displays differ in number and the size of each item is constant, then the more numerous display must have a larger cumulative area. Reprinted from Leibovich and Henik (2013, p. 2), Figure 1, in line with Frontier's CC BY License.

reliable on tasks of the sort under consideration. But it is not clear how this could be if those number judgments simply result from a sensitivity to non-numerical magnitudes (as skeptics maintain) and these are only insufficiently correlated with the numerical quantities in question. For example, if total surface area is sometimes positively, and other times negatively, correlated with numerical judgments, it is doubtful that numerical judgments (which are generally reliable) could simply be grounded in a sensitivity to total surface area.

4. The argument from confounds

The argument from congruency is unpersuasive. However, a more pressing objection concerns the fact that number cannot be presented to subjects independently of confounding variables. For instance, a visual display containing nine dots also contains dots with an average diameter, cumulative area, convex hull, and density. Similar points apply to heard or felt sets. Consequently, there has always been the worry that number isn't represented in studies of the above sort, only confounding variables. The *argument from confounds* claims that experimental attempts to evince an ANS with genuine numerical content are thereby undermined (Gebuis et al., 2016; Leibovich & Henik, 2013; Leibovich et al., 2017).

There are actually two readings of this argument. On a *strong* reading, it's deemed impossible to adjudicate between the hypothesis that subjects represent numerical quantities in addition to various non-numerical confounds and the hypothesis that they merely represent these confounds. By contrast, a *weak* reading of the objection holds that while it may not be impossible to distinguish these hypotheses, it's sufficiently difficult that there is currently no empirical justification to favor the number sense hypothesis over a leaner alternative.

We see no reason to accept the argument in its stronger incarnation. Theories in science are always underdetermined by the data, and the selection of one theory over another often requires an inference to the best explanation (Duhem, 1914). Therefore, in psychology, there may never be a single experiment that eliminates all potential confounds. This does not rule out a science of the mind, however. It simply invites us to consider multiple studies in tandem and to ask whether these better support one hypothesis over viable alternatives. The postulation of an ANS with number content is unexceptional in this regard. For while number is an abstract property that cannot be observed in isolation (Halberda, 2019), the same is true of many properties which are plausibly represented by the pre-linguistic mind, including *causation* (Kominsky & Carey, 2018) and *agency* (Gergely & Csibra, 2003). In each case, these hypotheses can be assessed against plausible alternatives. For to the extent that alternatives are undermined in controlled experiments, researchers can legitimately increase their credence in the relevant conjectures.

This leaves the weaker reading of the objection. To appreciate its force, consider studies that examine our visual perception of number by presenting arrays of dots on a screen. Some such studies choose one potential confound – say, total surface area – and keep it constant while number varies. Critics such as Leibovich et al. (2017, p. 4) correctly observe that this always leaves other confounds uncontrolled for (Fig. 3). For example, if total surface area is kept constant while number increases, average dot-size will need to decrease (see DeWind, Adams, Platt, & Brannon, 2015 for a precise characterization of these tradeoffs). Consequently, finding that subjects discriminate a difference fails to establish whether they are tracking number or average dot-size.

Other studies vary non-numerical magnitudes across trials, such that no one confound correlates with number throughout the whole experiment. Thus, half the trials might keep total surface area constant while the other half keep average dot-size constant (Dehaene, Izard, & Piazza, 2005; Halberda, Mazzocco, & Feigenson, 2008). Alternatively, each of a range of non-numerical magnitudes might be varied across trials such that, throughout the experiment, they are congruent on half of the trials and incongruent on the other half (Barth et al., 2005; Nys & Content, 2012). But, while these controls suggest that subjects do not rely on a single confounding magnitude, Gebuis et al. (2016, pp. 23–24) and Leibovich et al. (2017, pp. 4–6) object that subjects could still be switching between cues throughout the experiment or relying on multiple confounds. For instance, when total surface area is held constant or made incongruent with number, they might rely on average dot-size; and when average dot-size is held constant or made incongruent with number, they might rely on total surface area. In this way, subjects might behave as if they are sensitive to number when they are only sensitive to non-numerical confounds.

We accept that this is conceivable, but we deny that it provides reasonable grounds for doubt. This is because a plausible skepticism about the ANS cannot be *ad hoc*. It cannot rest on a piecemeal strategy of finding one set of confounds to account for behavior in one trial, a second set of confounds to account for behavior in another trial, and so on. What is needed is a positive proposal that explains how some function of confounds, or some principled strategy for switching between these, accounts for what appears to be number-tracking behavior across a wide range of trials and experiments. Skeptics of an ANS fail to provide one. Instead, they simply observe that numerical judgments are influenced by non-numerical magnitudes – that is, that they are subject to congruency effects. But, as we saw in section 3, these effects

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are fully compatible with the existence of an ANS that represents number.

To compound matters, experimental findings undermine the objection. Take the dumbbell effect discussed in section 2.1. As we explained then, connecting seen items with a thin line substantially reduces judgments of their number, while introducing a small break in these lines substantially reduces the effect (Franconeri et al., 2009; He et al., 2009; Kirjakovski & Matsumoto, 2016; see also Fornaciai & Park, 2018). Given that displays with and without a small break are nearly identical with respect to non-numerical confounds, these studies suggest that number is tracked and represented, and that performance is not simply a function of tracking non-numerical confounds as critics allege. We have found no discussion of these studies by those who remain skeptical of the ANS and its capacity to represent number.

Additionally, some studies reveal that the relevant sensitivity to number differs markedly from our sensitivity to non-numerical magnitudes. Indeed, DeWind et al. (2015) compared how number, size, and spacing of dots affect relevant numerical judgments and found that judgments were more sensitive to number than to size or spacing. This suggests that number itself is represented. (For a reply, see Leibovich et al., 2017, p. 10, and for a rebuttal to this see Park, DeWind, & Brannon, 2017 and Tomlinson, DeWind, & Brannon, 2020.) Similarly, Cicchini et al. (2016) had subjects judge the area, density, and number of dots in visual displays, and found that number judgments are more sensitive than area and density judgments. Again, this suggests that subjects do not simply represent area and density, but also numerical quantity (see also Odic, 2018; Savelkouls & Cordes, 2020).

Against this, it has been suggested that when perceived area is distinguished from mathematical area, number estimation is no longer shown to be more sensitive (Yousif & Keil, 2020). However, even here, the authors agree that "number estimation cannot be fully explained by perceived area" and that "the human visual system is certainly able to extract number." They only question whether number is being *directly* computed – an issue which is orthogonal to present concerns (see sect. 2.2).

Finally, we have already observed that cross-modal studies naturally eliminate potential confounds. As discussed in section 2.1, a static array of seen dots and a sequence of heard tones lack properties in common that could serve as a plausible crutch on which to base numerical comparisons. For while dots have a cumulative area, average diameter, and convex hull, tones have none of these properties. Because numerous studies demonstrate success in cross-modal tasks (Arrighi et al., 2014; Barth et al., 2003, 2005; Izard et al., 2009), this (again) undermines the argument from confounds.

Skeptics of the ANS do recognize this latter point. For instance, Leibovich et al. (2017) note that cross-modal studies provide "[a] very strong line of evidence supporting the ANS" (p. 5). But while they proceed to question whether cross-modal studies on infants show that the ANS is innate, we can bracket these worries because we aren't focusing on the issue of innateness. For our purposes, a more relevant response comes from Gebuis et al. (2016). They acknowledge the existence of number studies in human adults demonstrating cross-modal adaptation (Arrighi et al., 2014) and cross-modal comparison (Barth et al., 2003; Tokita & Ishiguchi, 2012). They also acknowledge that these bear the hallmark of ANS-based comparisons. In spite of this, they claim that such studies "do not present a clear result" (p. 27). They reason that if number were represented amodally, there should be no cost to cross-modal comparisons. But while the existence of some such cost remains under dispute (contrast: Barth et al., 2003; Gebuis et al., 2016; Tokita, Ashitani, & Ishiguchi, 2013), its discovery should not alarm proponents of an ANS. In intra-modal tasks, numerical comparisons are likely facilitated by confounding (typically, congruent) magnitudes (see sect. 3). Meanwhile, inter-modal tasks leave little opportunity for facilitation – after all, a static array of seen dots and a sequence of heard tones will lack properties in common that could serve to inform or bias numerical comparisons. In any case, it is the fact that cross-modal numerical comparisons are successfully executed at all that speaks in favor of an ANS.

In reply to all of this evidence, it might be argued that if indirect models of ANS computation are correct, then there has to be *some* description of what the ANS is doing that only appeals to non-numerical magnitudes. Therefore, given that we want to remain agnostic as between direct and indirect models (see sect. 2.2), what justifies us in holding that the ANS represents number rather than the mishmash of non-numerical magnitudes that would feature in that (hypothetical) description?

Earlier, we drew an analogy with depth, which is computed on the basis of diverse cues, such as binocular disparity, motion parallax, and accommodation. Even so, we said that people perceive depth, and not just these cues. It's important to appreciate why we said that. The reason is that *representation of depth* offers a unifying explanation that the motley of cues cannot. On one occasion, the visual system might rely primarily on binocular disparity to compute the depth of an object; on another occasion, it might rely primarily on motion parallax; on a third occasion, it might rely primarily on convergence; and so on. Appealing to a common representational kind in each case (*representation of depth*) allows us to provide a coherent, unified account of one's ability to judge depths in these and other circumstances, and it supports largely accurate predictions regarding the judgments you will make about depths on the basis of what you perceive. In short, *it supports generalizations*. It also helps explain why you are sensitive to these cues in the first place; that is, it helps explain what function they are serving. The hypothesis that the ANS represents number enjoys the same advantages over the hypothesis that the system merely represents a mishmash of non-numerical magnitudes to which a hypothetically adequate indirect model would appeal.

5. The argument from imprecision

This brings us to what is, perhaps, the most prominent critique of the ANS's capacity to represent number: *the argument from imprecision*. While this argument has a more philosophical flavor than the preceding objections, it has been repeated several times, often by scientists. We'll begin by outlining and critiquing a generic version of the argument, before examining two specific incarnations of the argument, due to Carey (2009) and Núñez (2017). Once again, we'll suggest that, as things stand, the argument fails to undermine the hypothesis that the ANS represents number.

5.1 The generic version

The argument from imprecision begins from the observation that the ANS is imprecise. When it processes a collection of 23 entities, say, it does not reliably produce a representation of exactly 23. If it did, we would have no difficulty discriminating collections of 23

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from collections of 22. But, given Weber's Law, we know that discrimination suffers as the ratio between magnitudes approaches 1:1. Thus, ANS representations must be imprecise in some way or another. But numbers themselves are not imprecise. The number 23 is *exactly one more than 22* and *exactly one less than 24*. The number 23 even differs from 23.00000000000001 by a precise amount (exactly 0.00000000000001). So (the argument goes), the ANS fails to represent numbers.

As stated, this argument is invalid. From the premise that the ANS represents imprecisely and the premise that numbers are precise it simply doesn't follow that the ANS fails to represent numbers. To illustrate, consider that non-numerical magnitudes such as distance, duration, and weight are also precise. There is a fact of the matter about exactly how heavy a given person is, and a fact of the matter about how much heavier/lighter one person is than another. But, while non-numerical magnitudes, such as weight, can be represented precisely, they needn't be. You can represent Jones as "240 pounds" or "as heavy as an NFL linebacker." In either case you are representing Jones' weight; what differs is how precisely you are doing so. *Prima facie*, the same is true of number. You can represent the number of coins in your pocket *precisely* as "exactly six" or *imprecisely* as "approximately six," or "several" (cf. Ball, 2017, p. 126).

Why, then, do so many theorists find the argument from imprecision compelling? The answer, we suspect, lies in a tacit commitment to *the sensitivity principle*. In broad strokes, the sensitivity principle holds that representing an entity requires sensitivity to its essential properties. But this principle can be read in a number of ways.

On a strong reading, it might be fleshed out as follows:

> *The Strong Sensitivity Principle:* if X has properties $p_1 \ldots p_n$ *essentially*, then representing X requires being sensitive to *all* of $p_1 \ldots p_n$.

If true, this would render the above argument valid. For if we were to grant that numbers are *essentially* precise, acceptance of this strong sensitivity principle would license the conclusion that a capacity to represent numbers requires sensitivity to this precision. Because this is something that the ANS does not provide, acceptance of the principle could thereby license the conclusion that the ANS fails to represent number.

The trouble is, this strong version of the sensitivity principle is false. To illustrate, note that while water is essentially H_2O, representing water does not require sensitivity to the property of being composed of two hydrogen atoms and an oxygen atom. You can think about how much you'd like a glass of water without knowing anything about the chemical composition of the water you desire. Similarly, gold is essentially the element with atomic number 79, but thinking that you'd like a gold watch does not require any sensitivity to atomic numbers. In fact, you needn't even be sensitive to the difference between gold and fool's gold. Many people who are ignorant of chemistry and metallurgy nevertheless think about water and gold (Burge, 1982; Putnam, 1975).

A similar point applies to perception. It seems to be an essential property of continuous magnitudes that they are dense; that between any two distances, durations, or weights, there is always a third. But, while perception represents these magnitudes, it has a limited resolution that prevents it being sensitive to their denseness. In representing motion, for example, the visual system relies on Reichardt detectors that cannot distinguish continuous from discrete changes in position (Green, 2018). That's why a string of Christmas lights that turn on in succession give rise to an illusion of motion (the *phi phenomenon*).

For this reason, proponents of the argument from imprecision must weaken their reading of the sensitivity principle. To this end, they might propose some version of the following:

> *The Weak Sensitivity Principle:* if X has properties $p_1 \ldots p_n$ essentially, then representing X requires being sensitive to *some* of $p_1 \ldots p_n$.

This weakened sensitivity principle enjoys *prima facie* plausibility. After all, it is perhaps plausible to suppose that the capacity to represent X requires sensitivity to *some* of X's essential properties even if it doesn't require sensitivity to *all* of these. For what else could make it the case that X is being represented rather than some other entity, Y?

The trouble is, even if this is granted, the weak sensitivity principle fails to fix the argument from imprecision. Even if representing numbers requires sensitivity to *some* essential properties of numbers, it doesn't follow that representing numbers requires sensitivity to the specific property of being precise. As a result, proponents of the argument from imprecision still require an (as yet unstated) reason for thinking that sensitivity to numerical precision is necessary to represent number.

Can a convincing reason be provided? While we can't rule it out, we have a hard time envisioning what it would look like.[1] Nor do we find any hints in the writing of those advocating the argument from imprecision. To illustrate, we will now consider two prominent incarnations of the argument, before turning to some positive reasons for thinking that the ANS represents numbers.

5.2 Carey on imprecision

Carey (2009, p. 295) invokes two versions of the argument from imprecision to maintain that the ANS is "not powerful enough to represent the natural numbers."[2] While these arguments have *prima facie* appeal – in fact, one of us endorsed them in earlier research (Beck, 2015) – they are unsound.

First, Carey contends that ANS representations "fail to capture small numerical differences between large sets of objects" (ibid., p. 294). For example, she writes that "the difference between eight and nine is not experienced at all" because eight and nine "cannot be discriminated" (ibid., p. 295). But, as Halberda (2016) points out, this erroneously presupposes that discriminability is binary. In fact, discriminability decreases smoothly as ratio increases, so there is no simple cut-off after which differences are not discriminated. In theory, even 999 and 1000 may be discriminated above chance given enough trials.

Furthermore, Carey assumes that if subjects aren't sufficiently sensitive to small differences between natural numbers (e.g., eight vs. nine), then they cannot represent natural numbers. But short of presupposing the problematically strong version of the sensitivity principle, it's unclear what justifies this assumption. The visual system represents distances even though it's relatively insensitive to small differences between these (e.g., 80 vs. 81 cm). This is because, there is nothing problematic in the thought that a precise distance (e.g., 80 cm) might be represented imprecisely. The same holds for number. To suggest otherwise is to mistake *what* the system represents (e.g., integers) for *how* it represents this (e.g., precisely or imprecisely).

Carey's second argument is that because the ANS treats five and six as more similar than four and five, it "obscure[s] the

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successor function," and thus cannot represent natural numbers (Carey, 2009, p. 295). Here, the suggestion is that a capacity to represent natural numbers requires a sensitivity to the successor function because the successor function is essential to natural numbers. But, again, this argument seems to presuppose the strong sensitivity principle, which should be rejected. For, short of providing some (as yet unspecified) reason for thinking that a sensitivity to the successor function is necessary to represent numbers, it fails to follow that a system (such as the ANS), which is insensitive to this, fails to represent number, even if it is granted that the successor function is an essential feature of number.

5.3 Núñez on imprecision

Núñez (2017) introduces his version of the argument from imprecision by noting that the ANS's numerical discriminations are "rarely exact" (p. 417) – they conform to Weber's Law. But, as he sees it:

> A basic competence involving, say, the number "eight," should require that the quantity is treated as being categorically different from "seven," and not merely treated as often – or highly likely to be – different from it. (ibid.)

Thus, Núñez proposes that the ascription of genuine numerical content to an ANS would require that it quantify "in an exact and discrete manner" lest this amount to nothing more than "loose" talk (p. 418). Because this is something that the ANS does not do, Núñez proposes that the ANS does not represent numerical quantities at all.

To be clear, Núñez is not proposing that the ANS is an *approximate* number system which represents numerical quantities inexactly. He is denying that it produces any numerical content whatsoever. This is evident in his "crucial distinction" between cognition that is "numerical" and cognition that is merely "quantical" (a term Núñez invents). Among other things, "quantical" cognition denotes "quantity-related capacities" that do not meet the requisite level of precision to qualify as genuinely numerical. Thus, Núñez urges that unless a system meets the requisite level of precision, it would be inappropriate to suppose it represents anything more than non-numerical quantities.

In saying this, Núñez lumps the ANS's representations with perceptual representations of other magnitudes, such as duration, brightness, distance, and chemical concentrations. All of them are on a par. They are all "quantical." But note that, in making this suggestion, Núñez fails to explain *why* a capacity to represent number requires a sensitivity to their precise nature. In this way, Núñez fails to address the challenge posed to proponents of the argument from imprecision in section 5.1.

On a charitable reading, it might be acknowledged that an appeal to "quantical" dimensions does, at least, offer an alternative to the view that the ANS represents numbers. Thus, it may seem attractive to those who are antecedently skeptical of the ANS's capacity to represent number. For this reason, it's important to stress that this is an unhelpful way of understanding matters. For one, an appeal to "quanticals" obscures a crucial distinction in this context. This is because, numerical quantities are higher order in that they can only be assigned relative to a sortal – a criterion for individuating the entities being counted (Frege, 1884). If we want to determine how many shoes are in your closet, it's not enough to open your closet and round up your shoes. We also need to decide whether we're counting individual shoes, pairs of shoes, or types of shoes. By contrast, if we want to know how much the shoes in your closet weigh, or what their total volume is, there is nothing further we need to do once we've identified the set of shoes. As Burge (2010, p. 472) notes, numerical quantities thus have a "second-order character" that non-numerical quantities lack. As such, it's important to recognize that the ANS represents properties with this second-order character even though its representations are imprecise.

To illustrate, recall the dumbbell studies by He et al. (2009) and Franconeri et al. (2009). When items are connected by thin lines, they are judged to be less numerous than when lines do not connect the items or contain small breaks. This indicates that the ANS takes a stand on how the entities in the array are individuated when they are being enumerated. Thus, it is not merely estimating the size of a first-order quantity. Rather, the ANS is sensitive to the second-order character of number. Because this second-order character is essential to number, it follows that the weak sensitivity principle is satisfied. There is at least one essential property of number to which the ANS is sensitive.

Appealing to the "quantical" obscures this feature of ANS representation because even first-order quantities qualify as "quantical" in Núñez's sense. To see why this should matter, consider a recent study by Plotnik et al. (2019). Here, elephants were presented with pairs of buckets containing sunflower seeds. These had opaque, perforated lids, allowing elephants to smell, but not see, their contents. Plotnik et al. found that elephants would preferentially select the bucket containing a greater quantity of sunflower seeds, albeit imprecisely and in accord with Weber's Law (Fig. 4). On this basis, they took their results to corroborate studies indicating the existence of an ANS in these creatures (e.g., Irie, Hiraiwa-Hasegawa, & Kutsukake, 2019). But note, while this might be so, it neglects a simpler possibility: Elephants were simply sensitive to the intensity of the odor emanating from the buckets, leading them to approach the bucket with the stronger odor (and hence more seeds). On this account, Plotnik et al.'s findings would be orthogonal to the presence or absence of an ANS with genuine numerical content; they would simply demonstrate these creatures' formidable capacity for olfaction.[3]

In studies with humans, non-numerical confounds have been carefully controlled for (see sect. 4). By contrast, Plotnik et al. fail to distinguish between the hypothesis that elephants represent number and the hypothesis that they represent odor. But it's a substantive question which is correct. And it's a question we would wish to answer whether or not the relevant discriminations are imprecise. This is because there's a basic distinction between representing a first-order magnitude, such as odor, and an ability to abstract away from this to represent a higher-order numerical magnitude. Núñez's approach obscures this important distinction because even odor representations are imprecise and quantitative, and thus satisfy his criteria for being "quantical."

6. Number versus numerosity

We have now considered three arguments which purport to undermine the conjecture that the number sense represents number and found no reason to reject this conjecture. We have also uncovered considerations that seem to support it. For instance, we have seen that ANS representations track second-order properties of concrete pluralities (albeit imprecisely) as opposed to first-order quantities. It should also be noted that ANS representations enter into arithmetic computations such as greater-than

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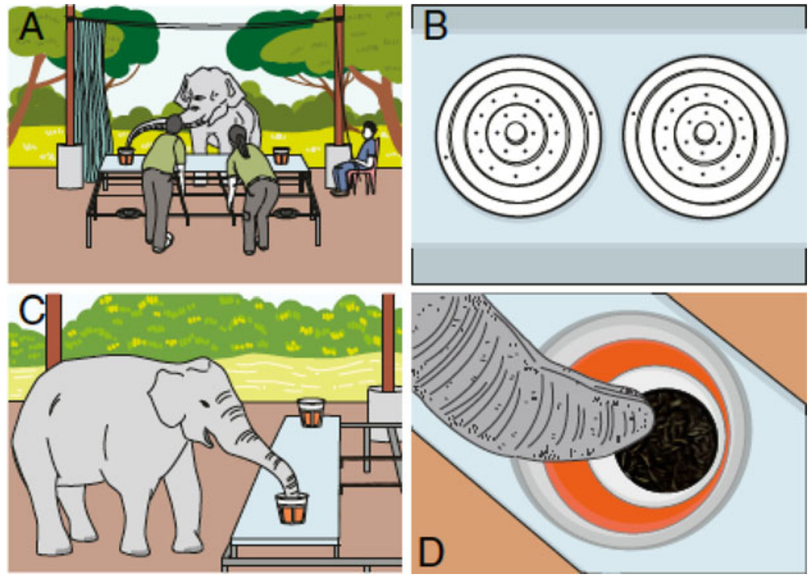

Figure 4. Elephants smelled two buckets containing sunflower seeds (a) with opaque, perforated lids (b). After they made their selection (c), the lid was removed, and they could eat the seeds (d). Elephants generally chose the bucket with a greater number of seeds, which (presumably) also had the more intense odor. Illustration by Nuttayapond Doungcharoen. Reprinted from Plotnik et al. (2019, p. 12567), Figure 1, in line with PNAS's licensing agreement.

and less-than comparisons, addition, subtraction, multiplication, and division (Barth et al., 2005, 2006; Barth, Baron, Spelke, & Carey, 2009; McCrink & Spelke, 2010, 2016; McCrink, Spelke, Dehaene, & Pica, 2012). Pending a convincing argument to the contrary, these considerations indicate that the ANS literally represents some kind of numerical magnitude.

Even so, it's crucial not to lose sight of the limitations inherent in the ANS that make its representations so different from the precise number concepts most human adults have at their disposal. There is a conceptual change that occurs when children acquire the capacity to count and properly use number words in their native language (Carey, 2009). There is also a deficit in human adults who retain an ANS but do not have access to precise number concepts – for example, because their community is bereft of natural number terms (Gordon, 2005; Pica, Lemer, Izard, & Dehaene, 2005), or because they suffer damage to their prefrontal cortex (Lemer, Dehaene, Spelke, & Cohen, 2003).

It is vital to acknowledge and mark this difference. But we should distinguish two ways of doing so. According to the first approach, these two types of representation differ because they represent different things. Whereas precise number concepts represent literal numbers (e.g., positive integers), imprecise ANS representations merely represent "numerosities," which are a kind of ersatz number. According to the second approach, these two types of representation differ not in *what* they represent – both literally represent numbers of a sort familiar from the math class – but in *how* they do so. That is, they have different modes of presentation. In the remainder of this section, we'll argue that this second approach is preferable.

It is notable that, throughout the ANS literature, the term "number" is assiduously avoided in favor of the neologism "numerosity." We suspect that this reflects an assumption that, strictly speaking, the ANS represents numerosities, not numbers. But what is a numerosity? Despite widespread employment of the term, ANS researchers almost never say. Apart from Núñez's appeal to "quantical" dimensions, which we criticized above, we know of only one other proposal that clearly distances numerosities from numbers: Burge's (2010) conjecture that these are "pure magnitudes."

Burge is skeptical that the ANS represents numbers but finds himself frustrated by psychologists' woolly use of the term "numerosity," calling it "a hedge term used to apply to number-like properties" (Burge, 2010, p. 472). To gain clarity, Burge recommends drawing inspiration from the ancient Greek theory of magnitudes attributed to Eudoxus and reported by Euclid.

According to Eudoxus, magnitudes divide into two species, continuous and discrete. Continuous magnitudes include length, weight, and duration. Discrete magnitudes include natural numbers. In addition, Eudoxus recognized the genus of these two species, *pure magnitudes*, which he used to account for (what we would now call) irrational numbers, such as $\sqrt{2}$. The Pythagoreans had noted that the diagonal of a square cannot be expressed as the ratio of two whole numbers. Eudoxus' theory of pure magnitudes was intended to provide a way to express such quantities. Burge speculates that the ANS represents pure magnitudes, which support basic arithmetic operations, but do not differentiate between discreteness and continuity. He writes, "I conjecture that the early Greeks articulated and formalized basic animal and childhood capacities when they theorized about magnitudes and ratios in a way that is unspecific as to whether the magnitudes are numbers or continuous quantities" (Burge, 2010, p. 483).[4]

The thing is, the ANS refers to a second-order property of collections (see sect. 5.2). It attributes a quantity relative to a way of sorting or individuating particulars. But, as the genus of discrete and continuous magnitudes, pure magnitudes do not differentiate between magnitudes that have a second-order character (e.g., natural numbers) and magnitudes that lack such a second-order character (e.g., length). Pure magnitudes are, thus, poorly suited to capturing the contents of ANS representations.

But if numerosities aren't pure magnitudes then what are they? The simple fact is, no one has any idea. And that seems like good reason to avoid positing numerosities as the contents of ANS representations. Better to say that the ANS represents numbers of a familiar sort.[5]

In saying this, we avoid a curious double standard that plagues discussions of the ANS. For, as we have already noted, perceptual representations of non-numerical magnitudes, such as distance, duration, and weight, are also imprecise and governed by Weber's Law. But we have not come across a single passage which concludes that we thereby represent "distancosity," "durationosity," or "weightosity." And for good reason.

Although no one knows what a distancosity, durationosity, or weightosity would be, distance, duration, and weight are respectable and familiar entities in our scientific ontology. The imprecision inherent in our discrimination of non-numerical magnitudes is thus not taken to prevent distance, duration, or weight from being represented; it's merely taken to modify the way in which they are represented. Taking the ANS to represent number rather than numerosity allows for greater consistency with our treatment of non-numerical magnitudes.

At this point, it would be nice if we could support our proposal with a naturalized theory of content that explains why the ANS represents number rather than numerosity. But we don't have one, and the leading theories on offer (e.g., Neander, 2017; Shea, 2018) aren't pitched at the right level of granularity. This being said, our search for the referent of a representation should be biased toward entities we have independent reason to posit in our scientifically informed ontology. Among other virtues, this allows psychological explanations invoking representational content to be integrated with explanations from other sciences, such as biology. We are, thus, in agreement with Burge when he counsels that the visual system of the frog should be taken to represent flies rather than undetached fly parts because undetached fly parts "are not kinds that ground biological explanations of the frog's needs and activities" (Burge, 2010, p. 322; see also p. 466). Since biological explanations appeal to numbers, not numerosities, we find reason to favor the conjecture that numbers are the referents of ANS representations, and not numerosities as others propose.

Finally, it bears emphasizing that there is no need to invoke peculiar entities such as numerosities to make sense of differences between ANS representations and precise number concepts. These differences can simply be captured by appeal to their modes of presentation. As Frege (1892) observed, two names can refer to the same object even though they differ in what we intuitively think of as their meanings. Thus, while both "Hesperus" and "Phosphorus" refer to Venus, they have different modes of presentation. It was a substantive empirical discovery that the names co-refer. And what goes for names, goes for mental representations (Burge, 2005; Dummett, 1981; Evans, 1982; Peacocke, 1992). This is clearly true in thought – the Babylonians' concepts *Hesperus* and *Phosphorus* had distinct modes of presentation. Similarly, modes of presentation differ between perception and thought. As Peacocke (1986) noted, knowing the length of a piano in feet and inches may not settle whether it will fit along a wall in your living room even if you're looking straight at the wall. The reason is not that the property you entertain in thought differs from the property you entertain in perception, but that you entertain the property in different ways – that is, under different modes of presentation. This is also the most natural thing to say about how ANS representations differ from precise number concepts.

In fact, there is independent reason to think that ANS representations and precise number concepts differ substantially in their mode of presentation. That is because the representations involved have different formats. The fact that the ANS obeys Weber's Law indicates that its representations are analog, like a mercury thermometer (Beck, 2015, 2018, 2019; Clarke, forthcoming). Precise number concepts, by contrast, have a non-analog, language-like format. Differences in format spawn differences in mode of presentation (Beck, 2013) but are compatible with sameness of reference. Just as both an analog watch and a digital watch represent time, both the ANS and our precise number concepts represent number – albeit in different ways.

In saying this, we do not assume that precise number concepts are ontogenetically grounded in the ANS. For example, we do not claim that learners acquire precise number concepts by mapping them onto their ANS representations. In fact, there are reasons to be skeptical of this claim (Carey & Barner, 2019). Such reasons largely derive from the mismatch in format and precision between ANS representations and precise number concepts. But, as we've seen, that mismatch in format and precision does not require a mismatch of reference; on the contrary, it is better characterized by a mismatch in mode of presentation.

This is important to recognize because researchers sometimes suggest that the ANS can directly ground precise number concepts if, and only if, it represents numbers. For example:

> if the ANS represents numbers, then its representations can directly link to our processing of symbolic numbers. If something like pure magnitudes… are represented, then the ANS has to bridge a gap to be relevant to symbolic number tasks. (Buijsman, 2021, p. 304)

But this is a mistake. A gap needs to be bridged whether or not the ANS represents numbers.

Are we just playing a linguistic shell game? Is there a substantive difference between saying that the ANS represents numerosity (not number) and saying that it represents number but with a different mode of presentation than precise number concepts?

The difference is substantive. There is a clear difference between systems that represent distinct quantities, such as a clock and a thermometer, and systems that represent the same quantity in different ways, such as a digital clock and an analog clock. Researchers who take the ANS to represent non-numerical or faux-numerical quantities are simply mischaracterizing the ANS. This has practical implications, such as a tendency to overlook the second-order character of ANS representations and to underestimate what it takes to tap the ANS itself (as in Plotnik et al., 2019). It also leads number researchers on a fool's errand – searching for, and attempting to illuminate, mysterious properties, such as "quanticals." But, if we are right that the ANS represents number, researchers can halt the wild goose chase and refocus more fruitfully on factors that contribute to the unique mode of presentation found in ANS representations, such as their format, imprecision, and computational role.

Strange as it is to have to say: the number sense represents number.

7. What kind(s) of number?

We have now argued that the ANS represents numbers of a familiar variety: Extant arguments to the contrary are unpersuasive (sects 3–5) and there is positive reason to endorse the conjecture (sect. 6). But this raises the neglected question noted in section 1: numbers of what kind?

Before proceeding, we should stress that there are various things one might be asking with this question. So, to clarify, we'll be asking how *fine-grained* the numbers represented by the ANS are. Thus, we'll be asking whether the ANS represents natural numbers, which are relatively coarse-grained (sect. 7.1), real numbers, which are extremely fine-grained (sect. 7.2), or rational numbers, which have an intermediate grain (sect. 7.3). In so doing, we'll remain neutral on whether the ANS is committed to a specific axiomatization, analysis, or ontology of the numbers it represents. We'll also remain agnostic about whether we should think of the ANS as representing precise numbers

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(7, 1.5, etc.), precise numbers with a confidence estimation attached (Halberda, 2016), numerical intervals (5–9, 1.25–1.75, etc.) (Ball, 2017), or probability distributions over numerical intervals. While interesting and important, these questions are orthogonal to whether the ANS represents natural, real, or rational numbers. Since the question of whether the ANS represents natural, real, or rational numbers already bears on the computations an ANS performs and facilitates, answering this question would constitute a substantial contribution to our understanding of the ANS.

7.1 Natural numbers

To begin, let's consider the conjecture that the ANS represents natural numbers, or *positive integers*. While rarely defended (but see Ball, 2017), this conjecture seems to be assumed by those who maintain that precise integer concepts derive from the ANS (Dehaene, 2011; Nieder, 2017; Park & Brannon, 2013; Piazza, 2011; Starr, Libertus, & Brannon, 2013; Wagner & Johnson, 2011). Indeed, once one concedes that the ANS represents numbers, the conjecture that the ANS represents natural numbers may seem to constitute the default hypothesis. On this view, the ANS functions to represent whole numbers, such as 7 and 8, and not comparatively esoteric numbers, such as $\sqrt{2}$, whose initial identification constituted major mathematical discoveries.

While we are skeptical that the ANS grounds precise integer concepts (Carey & Barner, 2019), there is strong reason to think the ANS does represent positive integers. We say this because, as we have seen, the ANS functions to keep count of *whole items*, at least in paradigm cases. Exactly what these items might be varies from stimulus to stimulus, or even as a result of shifting attention. Thus, the ANS might represent the number of visual objects in an array, or objects of a certain type (such as closed shapes, or circles, or red circles as opposed to green ones, and so forth). Or, rather than representing material objects, it might represent a number of events, such as rabbit jumps (Wood & Spelke, 2005), or auditory items in a sequence, such as tones or phonemes heard. In each case, the ANS functions to represent the number of *whole items* in the relevant set. Because the number of whole items in a given set is expressed by a positive integer, we conclude that, at a minimum, the ANS represents positive integers.

7.2 Real numbers

In one of the few discussions to address the kinds of number the ANS represents, Gallistel and Gelman (2000) defend a striking hypothesis: The ANS goes beyond representing natural numbers by representing *real numbers*, which include not only integers, but also rational numbers, which can be expressed as a ratio of integers (e.g., 1.5), and irrational numbers, which cannot be so expressed (e.g., $\sqrt{2}$). In fact, Gallistel and Gelman conjecture that the ANS *primarily* represents irrational numbers, because "all but a negligible fraction of [the real numbers] are irrational" (p. 59). This is a remarkable result. Is it correct?

If the ANS represents irrational numbers then, all things being equal, we would expect this to be manifested in ANS-governed behavior. But Gallistel and Gelman do not point to any behavioral evidence of this sort; nor do we know of any. For while there is ample evidence that the ANS represents whole numbers, we know of no evidence that it represents $\sqrt{2}$, say. As it stands, the hypothesis that the ANS represents irrational numbers would seem unsupported by existing evidence.

Why, then, do Gallistel and Gelman endorse the hypothesis? They reason as follows. First, because duration is a continuous magnitude, they infer that it cannot be represented by anything discrete. Therefore, the neural magnitude that represents duration must be continuous. Next, drawing on evidence in rats (Church & Meck, 1984), they infer that numbers are represented by neural "magnitudes indistinguishable from those which represent duration" (Gallistel & Gelman, 2000, p. 62). Thus, the ANS must also use a continuous neural magnitude. Finally, because real numbers are continuous, but integers are not, they conclude that the ANS must represent real numbers rather than just integers.

This argument is problematic on various fronts. First, while duration is a continuous magnitude, it is unclear whether duration's continuity is reflected in the grain of duration representations. Just as there is a dearth of evidence that the ANS represents irrational numbers, such as $\sqrt{2}$, there is a dearth of evidence that durations such as $\sqrt{2}$ s are represented by comparable mechanisms. So, while duration representations may be very fine-grained, there is no evidence that they are genuinely continuous (Laurence & Margolis, 2005, p. 223, n. 7). Gallistel and Gelman may be led to think otherwise because they take Weber's Law to be evidence of vehicles that are continuous neural magnitudes. But, so long as the right type of noise is introduced, Weber's Law can be explained by discrete neural vehicles, such as the number of neurons firing above some threshold within a given population (Beck, 2015; Maley, 2011).

Second, Gallistel and Gelman's argument rests on the mistaken assumption that the content of a representation mirrors its format. Thus, they assume that if duration is represented as continuous, then the vehicle employed must be continuous too; and if the vehicles employed by the ANS are continuous, then the content of the ANS is a continuous number line (reflecting the real numbers). But, as Laurence and Margolis (2005) point out in a compelling critique of Gallistel and Gelman's argument, we use discrete vehicles to represent continuous contents all the time. For example, discrete symbols such as "π" and "$\sqrt{2}$" express precise irrational numbers. Conversely, digital computers use continuous magnitudes, such as voltage, to represent discrete values (Lewis, 1971; von Neumann, 1958). Thus, "There is nothing at all incoherent about mental magnitudes representing discrete values" (Laurence & Margolis, 2005, p. 224).

Given the shortcomings in Gallistel and Gelman's argument, and the general lack of empirical evidence for their conclusion, the hypothesis that the ANS represents irrational numbers should be rejected pending a convincing argument to the contrary.

7.3 Rational numbers

At this point, it may be tempting to suppose the ANS merely represents natural numbers. After all, there is compelling reason to think that the system does, in fact, represent natural numbers, and the most prominent account on which the ANS goes beyond representing these is unpersuasive. Against this, we recommend an intermediate position. On our view, the ANS goes beyond representing natural numbers (e.g., 7) by representing (non-natural) *rational* numbers (e.g., 3.5). That is because the ANS represents *ratios* among positive integers, in addition to positive integers themselves, and rational numbers are expressible as ratios among positive integers. While we do not take this hypothesis

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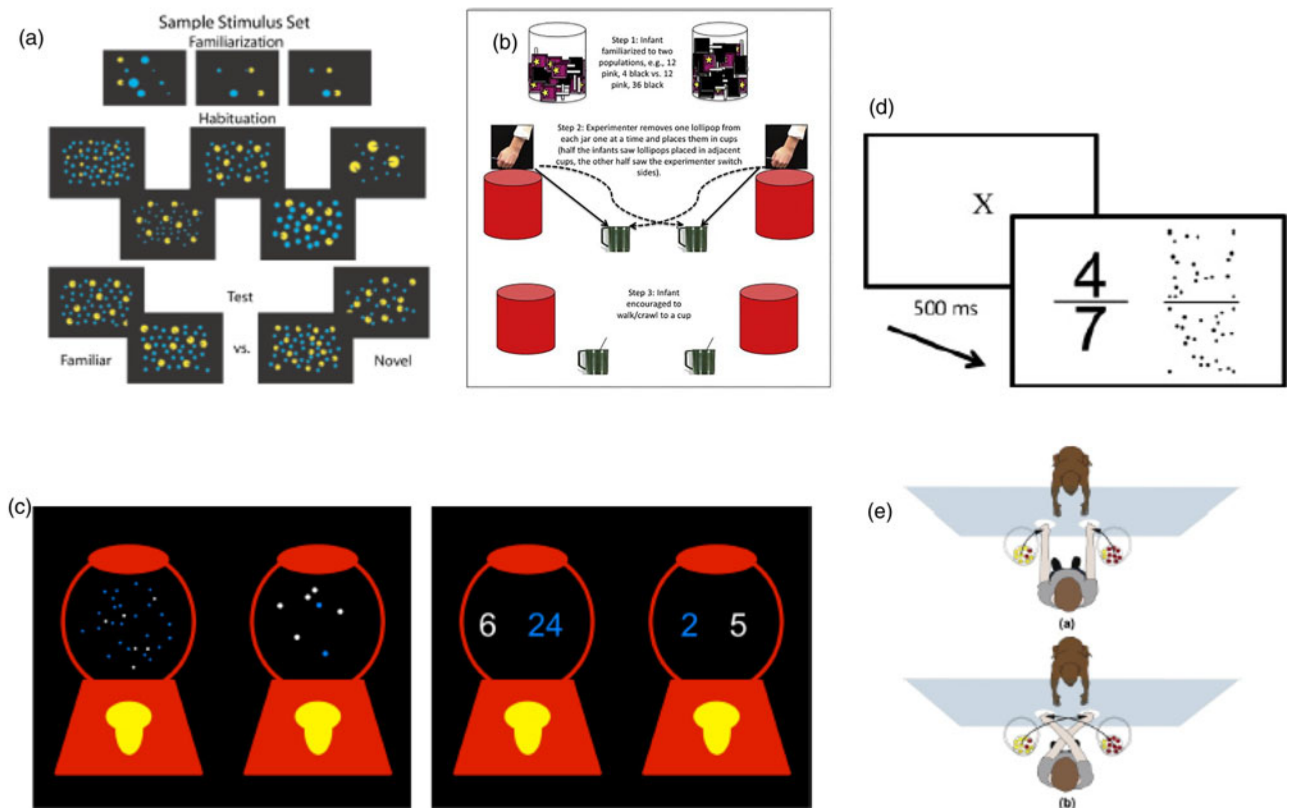

Figure 5. (a) Habituation stimuli in the figure illustrate the 4:1 ratio that infants were habituated to. Reprinted from McCrink and Wynn (2007, p. 742), Figure 1, Copyright © 2007, with permission from Sage Publications. (b) Test condition. Reprinted from Denison and Xu (2014, p. 338), Figure 1, Copyright © 2014, with permission from Elsevier. (c) Illustration of the non-symbolic (left) and symbolic (right) ratio comparison tests. Reprinted from Szkudlarek and Brannon (2021, p. 3), Figure 1, Copyright © 2021, with permission from the Society for Research in Child Development. (d) Example of a symbolic and non-symbolic fraction comparison. Reprinted from Matthews and Chesney (2015, p. 33), Figure 1, Copyright © 2015, with permission from Elsevier. (e) Rakoczy et al.'s study mirror's Denison and Xu's (2014), but with nonhuman apes instead of infants. Reprinted from Rakoczy et al. (2014, p. 63), Figure 1, Copyright © 2014, with permission from Elsevier.

to be definitively established, we consider it well-motivated and strongly suggested by a range of empirical findings. Moving forward, it should be our working hypothesis.

The proposal we are making is closely related to a suggestion from the developmental and educational psychology literatures according to which there is a "ratio processing system" (RPS) which stands to our understanding of fractions in the way a number sense has traditionally been seen to stand to whole number understanding (Bhatia et al., 2020; Binzak & Hubbard, 2020; Lewis, Matthews, & Hubbard, 2016; Matthews & Chesney, 2015; Matthews, Lewis, & Hubbard, 2016; Siegler, Fazio, Bailey, & Zhou, 2013). That being said, it isn't always clear whether the RPS is supposed to be a component of the ANS (as we'll suggest) or a separate system. Moreover, the hypothesis that the RPS represents rational numbers is not always clearly distinguished from the conjecture that it represents real numbers more generally. Indeed, it is sometimes assumed that evidence for the former hypothesis would be a stepping stone toward vindicating the latter conjecture (e.g., Matthews et al., 2016, p. 191).

Consequently, it is worth considering why one might expect a system such as the ANS to represent rational numbers, and not just natural numbers. The reason, we suggest, is that organisms need to reason under uncertainty. They need to draw inferences (e.g., about the future) from limited data. The capacity to engage in fast and efficient probabilistic reasoning would, thus, be enormously advantageous (Tenenbaum, Kemp, Griffiths, & Goodman, 2011). And in fact, a flood of research suggests that even human infants engage in probabilistic reasoning of this sort (Denison & Xu, 2014; Girotto & Gonzalez, 2008; Gweon, Tenenbaum, & Schulz, 2010; Kayhan, Gredebäck, & Lindskog, 2018; Téglás et al., 2011; Xu & Denison, 2009; Xu & Garcia, 2008). To do so, they need to represent probabilities; and the most straightforward way to do this is to represent rational numbers.

In one study to probe representations of this sort, McCrink and Wynn (2007) habituated 6-month-old infants to multiple examples of a single numerical ratio and found that they would subsequently look longer when presented with a novel ratio (Fig. 5a). For instance, infants habituated to displays containing a 4:1 ratio of blue pellets to yellow Pacman shapes recovered interest when presented with a display containing a 2:1 ratio. Because the number of individual pellets and Pacman shapes was varied across habituation displays (while keeping the ratio between these constant and other confounds controlled for) this suggests that infants locked onto, and subsequently responded to, a change in *the numerical ratio* between these elements, abstracting away from the specific number of pellets and/or Pacman shapes presented in each display. Furthermore, just as Xu and Spelke (2000) found that 6-month-old infants could reliably discriminate absolute numerical values in a 2:1 ratio but not a 3:2 ratio, McCrink and Wynn (2007) found that they could reliably discriminate the ratio 4:1 from 2:1 but not the ratio 3:1 from 2:1. This similarity in performance is suggestive of a shared system.

Using a different approach, Denison and Xu (2014) tested ratio understanding in 10- to 12-month-old infants (Fig. 5b). They presented each infant with two lollipops, one pink and one black, to see which they preferred. They then showed the infants two transparent jars containing pink and black lollipops in different ratios and sampled from these populations by placing a lollipop from each jar in an opaque cup without allowing the infant to see its color. The infants reliably walked or crawled toward the sample that came from the jar with the higher ratio of their preferred lollipop. Crucially, this persisted even when the jar with the higher ratio of preferred lollipops contained fewer preferred lollipops in absolute terms. Thus, for pink-preferring infants, when one jar contained 16 pink and 4 black lollipops, and the other contained 24 pink and 96 black lollipops, they approached the sample drawn from the first jar rather than the second. (See also Denison & Xu, 2010; Fontanari, Gonzalez, Vallortigara, & Girotto, 2014; Kayhan et al., 2018; Xu & Denison, 2009; Xu & Garcia, 2008.)[6]

One limitation of this lollipop task is that infants could have succeeded by representing ratios of (pink vs. black) surface areas rather than numbers. But, while Denison and Xu (2014) didn't control for area, McCrink and Wynn (2007) did. Studies with older children also control for area and other non-numerical quantities. For example, Szkudlarek and Brannon (2021) presented 6- to 8-year-olds with pairs of depicted gumball machines (Fig. 5c). Each gumball machine was filled with either blue and white "gumballs" (dots) or blue and white Arabic numerals specifying the number of blue and white gumballs contained therein. In each case, the children were tasked with selecting the gumball machine with the best chance of producing a gumball of a desired color, taking into account the ratio of blue to white gumballs contained therein. They were able to do so reliably in both non-symbolic (dot) and symbolic (numeral) conditions, even though children of this age have not yet begun to study fractions in school and struggle with precise fraction comparisons. Because subsequent analyses indicated that the children's performance could not be attributed to simpler heuristics (e.g., choosing the machine with *more* desired gumballs, or the machine with *fewer* undesired gumballs), these results suggest that 6–8-year-olds represent numerical ratios. And because this capacity was manifested in symbolic displays, where the numerator and denominator were specified with Arabic numerals, non-numerical confounds were largely eliminated.

Matthews and Chesney (2015) conducted a related study in which college students were tasked with choosing the larger fraction when symbolic arrays (e.g., $^4/_7$) were pitted against discrete non-symbolic arrays in which the numerator and denominator were expressed with dots, and again when these discrete non-symbolic arrays were pitted against continuous non-symbolic arrays in which the numerator and denominator were each replaced by a circle of variable area (Fig. 5d). Subjects succeeded in these comparisons even though they answered too quickly to have explicitly counted the dots in the discrete non-symbolic arrays. Importantly, their reaction times and errors were predicted by Weber's Law, suggesting that they used their ANS to represent ratios among whole numbers.

Finally, just as there is evidence that nonhuman animals discriminate among absolute numerical values, there is evidence that monkeys (Drucker, Rossa, & Brannon, 2016; Tecwyn, Denison, Messer, & Buchsbaum, 2017) and nonhuman apes (Eckert, Call, Hermes, Herrmann, & Rakoczy, 2018; Rakoczy et al., 2014) discriminate numerical ratios (Fig. 5e).

Given converging evidence that the ANS supports comparisons among ratios of positive integers, we conclude that the hypothesis that the ANS represents rational numbers deserves to be provisionally endorsed. But note two things. First, this conjecture is still pitched at a *computational level of analysis* (sect. 2.2). Thus, it is not wedded to a specific account of the system's underlying architecture. For this reason, it is a further question to what extent (if any) the neural and/or psychological mechanisms involved in the ANS's representation of natural and rational numbers overlap. To clarify, note that *the visual system* is often viewed as unified by its function, despite comprising relatively autonomous sub-modules performing dedicated tasks at various levels of visual analysis (Clarke, 2021; Fodor, 1983; Marr, 1982). Analogously, our suggestion that the ANS represents both natural and rational numbers allows us to remain neutral on the (important but additional) question of whether it comprises autonomous components dedicated to natural number representation, on the one hand, and rational number representation (the RPS proper), on the other. This is because, even if these components are distinct, and even largely encapsulated from one another, the ANS (as we have understood it) remains unified on account of its unified functional profile: representing numbers in accord with Weber's Law.

Second, our conjecture that the ANS represents rational numbers does not commit us to claiming that the system represents *every* rational number or even *most* rational numbers. This bears emphasizing because the conjecture has previously been dismissed on these grounds. For instance, Marshall (2017, p. 49) claims that the ANS cannot represent the rational numbers because the rational numbers are dense – between any two rational numbers there is always a third – and the ANS does not respect this feature. For while the ANS probably represents 2.5 and 2.75, there is no evidence that the ANS can represent 2.7452294861. This objection should, however, have a familiar ring: It seems to presuppose the strong sensitivity principle, which (as we've seen) must be rejected. Just because rational numbers are essentially dense doesn't mean that the ANS must be sensitive to their denseness to represent them. Just as the ANS can represent natural numbers such as 7 even though it cannot represent all of the natural numbers (e.g., 1 trillion is surely beyond its upper limit), the ANS can represent positive rational numbers such as 2.5 even if it cannot represent all positive rational numbers.

Earlier, we argued that the ANS represents numbers, not non-numerical magnitudes, because it is sensitive to the second-order character of numbers, which is an essential property of numbers but not non-numerical magnitudes. Now we're arguing that it represents rational numbers, not just integers, because it's also sensitive to ratios, which are an essential property of rational numbers but not integers. Note, however, that there are no essential properties of irrational numbers, but not rational numbers, to which the ANS is sensitive (so far as we can tell). Thus, there is no parallel reason to say that the ANS represents real numbers more generally. The train terminates at rational numbers.

If the ANS can represent rational numbers, what would prevent it from representing irrationals? We suspect that this may be a byproduct of how it operates, privileging the representation of whole entities. After all, it is plausible that *all* of the aforementioned studies involved the ANS, first, representing natural numbers of concrete pluralities, and only then deriving ratios (hence,

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rational numbers) therefrom. For instance, in McCrink and Wynn's (2007) study it is natural to suppose that the system, first, represented natural numbers of pellets and Pacman shapes, and only then contrasted these so as to identify the ratio between them. But, because there is no way to represent irrational numbers as a ratio of natural numbers, the existence of some such process could not undergird the representation of real numbers more generally.

Of course, much research remains to be carried out developing and testing our hypothesis. The study of ratio understanding is still relatively young. The aforementioned studies are vulnerable to defeater explanations – for example, pitched in terms of a mere sensitivity to ratios between non-numerical confounds such as area. Thus, evidence that the ANS represents rational numbers remains provisional. But psychologists could adapt paradigms which discredit confound-based explanations for natural number tracking (sect. 4), such as cross-modal comparisons and dumbbell stimuli, to adjudicate these concerns. It would also be nice to see studies that more directly test whether identical ratios (e.g., 4:8 and 16:32) are treated as such.

Our skepticism surrounding the ANS's ability to represent *irrational* numbers stems (in large part) from the lack of positive evidence to support this proposal. But, considerations of parsimony to one side, absence of evidence is not evidence of absence. Thus, scientists might seek out empirical evidence that irrational numbers (e.g., π) feature in the computations the ANS performs. For instance, they might consider the ANS's potential involvement in calculating square roots or logarithms. Of course, care must be taken to ensure that the ANS is being tested and not some non-numerical magnitude system. For example, the ability to compute a circle's area from its radius would not require computations over *any* numbers, let alone irrational numbers, if non-numerical magnitudes such as length and area are represented in a unit-free manner (Peacocke, 1986). Moreover, even if these computations did implicate numerical magnitudes, careful controls/arguments would be needed to show that they involve a representation of irrational numbers and not rational approximations thereof. Regardless, we believe that reflection on these cases may help empirically distinguish our proposal from that advanced by Gallistel and Gelman.

8. Conclusion

We have argued that the ANS represents numbers of a familiar sort, and tentatively suggested that this involves it representing both positive integers and rational numbers, but not the reals more generally. We have drawn this conclusion because arguments to the contrary are flawed (sects 3–5), because the postulation of genuine number content has theoretical and explanatory advantages over the postulation of alternatives such as "quanticals," "pure magnitudes," or "numerosities" (sect. 6), and because the conjecture that these contents include natural and rational (but not irrational) numbers makes best sense of the existing data (sect. 7). In so doing, we hope to have quelled recent skepticism surrounding the ANS's ability to represent number, clarified the nature of its representations, and highlighted fruitful questions to be investigated in future research.

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Notes

1. One possibility would be to maintain that representing X requires being sensitive to those properties that feature in X's definition, or "individuation conditions." Thus, if it is definitive of numbers that they are precise, then representing numbers would require sensitivity to their precision (cf. Peacocke, 2020, pp. 154–156). But that approach requires a highly controversial commitment to definitions (Fodor, 1998; Quine, 1951), which proponents of the argument from imprecision do not defend. Moreover, the most plausible versions of this principle are restricted to conceptual representation under a canonical mode of presentation (Peacocke, 2020). (To *canonically* represent X, one must be sensitive to the properties that are definitive of X.) But we are primarily concerned with specifying the *referents* of ANS representations, not their modes of presentation. Thus, while we maintain that the ANS represents numbers, we are happy to concede that it does so non-canonically and non-conceptually.

2. Although Carey denies that the ANS represents natural numbers, she occasionally says it represents cardinal numbers (e.g., Carey, 2009, p. 136). As Burge (2010, p. 480, n. 82) observes, these claims are hard to square. (The natural numbers are often identified with the finite cardinals.) Ball (2017, p. 135, n. 9) attempts to resolve the tension, proposing that while natural numbers are best defined in terms of the successor relation, cardinal numbers are best defined in terms of Hume's principle (that the *F*s and *G*s are equinumerous if and only if there is a one-to-one correspondence between them). Thus (according to Ball), whereas representing natural numbers requires sensitivity to the successor relation, representing cardinal numbers requires sensitivity to Hume's principle. Carey may then maintain that the ANS represents cardinal numbers but not natural numbers if she maintains that the ANS is sensitive to Hume's principle but not the successor relation. We deny that such sensitivity is required in either case. But we also don't see the justification for claiming that the ANS is sensitive to Hume's principle. Thus, we think the tension Burge identifies in Carey's exposition remains.

3. Our worry here is very different from that which underpins the argument from confounds. Our worry is that Plotnik et al. failed to control for a single, specific capacity that we have independent reason to attribute to elephants. By contrast, the worry that motivates the argument from confounds is that there could be some unspecified and gerrymandered mix of cues that experimenters fail to control for.

4. Buijsman (2021) endorses Burge's suggestion, and supplements it with an account of indeterminate vehicles to explain the ANS's imprecision. Buijsman (2021, p. 310) acknowledges that readers might wonder why he says that the ANS represents pure magnitudes rather than natural numbers, and replies that natural numbers "cannot be indeterminate" because, "There are no alternative choices for '1' as the unit value of the natural numbers which are equally good, whereas there are alternative choices for '1 cm' which are equally good, namely 1 inch, 1 meter, and so on." We doubt that 1 is the only "equally good" unit of measurement for natural numbers. (Is it a shanda to buy bagels by the dozen?) But, even if it is, and even if individual natural numbers cannot be indeterminate, ranges of natural numbers can be. Therefore, we still don't see why indeterminacy favors the hypothesis that the ANS represents pure magnitudes over the hypothesis that it represents (ranges of) numbers.

5. A reviewer suggests that number researchers may be reticent to jettison the term "numerosity" because it's useful to refer to a concrete plurality – for

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example, a collection of six dots on a screen. But one can represent a concrete plurality without representing its numerical value, and what is distinctive of the ANS is not merely that it represents concrete pluralities, but that it attributes numerical values to them. Another reviewer wonders what researchers who are accustomed to the term "numerosity representation" should use instead to avoid confusion with precise number concepts. Where context isn't sufficient, we recommend "ANS representation," "approximate number representation," or "analog number representation." By contrast, the precise number concepts acquired later in development can simply be called "precise number concepts," "acquired number concepts," or "conceptual number representations."

6. One issue that isn't settled by this study – or, to our knowledge, any other extant study – is whether infants succeed by using ratios or fractions. Do they choose the jar that contains 16 pink lollipops because the ratio of 16 pink to 4 black is more favorable than the ratio of 24 pink to 96 black? Or, do they choose it because the ratio of 16 pink to 20 total lollipops is more favorable than the ratio of 24 pink to 120 total lollipops? Only the latter ratios are equivalent to fractions – that is, 4/5 vs. 1/5. Adding additional colors and then testing for a cost (in terms of errors or reaction time) might provide some insight. But, either way, what is represented goes beyond mere integers, and thus seems to require an appeal to rational numbers.

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Open Peer Commentary

Perceived number is not abstract

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Abstract

To support the claim that the approximate number system (ANS) represents rational numbers, Clarke and Beck (C&B) argue that number perception is abstract and characterized by a second-order character. However, converging evidence from visual illusions and psychophysics suggests that perceived number is not abstract, but rather, is perceptually interdependent with other magnitudes. Moreover, number, as a concept, is second-order, but number, as a percept, is not.

The concept "seven" applies just as easily to seven *elephants*, as seven *mice*, as seven *apples*. Numbers, as concepts, are abstract entities. "Seven" can be used to describe sets that differ vastly in their perceptual makeup because the symbol "seven" is dissociated from the sensory input (Dehaene, 1992). In making the case for non-symbolic number as a genuine dimension, Clarke and Beck (C&B) argue that the approximate number system (ANS), a perceptual system, is similarly abstract. Thus, whether elephants, mice, or apples, the ANS represents number, irrespective of their physical differences. In support of this perspective, C&B argue that (1) number is independent of other magnitudes; and (2) number, unlike other magnitudes, exhibits a "second-order" property. Together, these arguments form the prerequisites for rational number.

Here, we argue that number is not (perceptually) independent of other magnitudes, nor is it unique by comparison. Moreover, we suggest that the appeal to a second-order character for number fundamentally confuses the distinction between percept and concept, such that number, as a percept (and similar to other magnitudes) is not second order.

C&B argue that congruency effects do not dispute number as an abstract entity because the interactions between number and other magnitudes reflect post-perceptual processes such as a response-stage conflict. However, other evidence, not cited by C&B, provides more definitive evidence for the non-independence of number from other magnitudes. For example, by making clever use of Müller-Lyer (Dormal, Larigaldie, Lefèvre, Pesenti, & Andres, 2018) and Ebbinghaus (Picon, Dramkin, & Odic, 2019) illusions, researchers have demonstrated that number perception is influenced by illusory changes in non-numerical magnitudes such as length and density. Importantly, such effects were observed with estimation tasks in which participants estimated number, ruling out a response-stage conflict typical of magnitude comparison tasks, in which the response applies to both task-relevant and task-irrelevant magnitudes.

Our own research suggests that number and area are perceived holistically as integral dimensions (Aulet & Lourenco, 2021a). In Aulet and Lourenco, we found that perceived similarity for dot arrays, which varied parametrically in number and cumulative area, was best modeled by Euclidean, as opposed to city-block, distance within the stimulus space (Garner, 1974; Shepard, 1964), comparable to classically integral dimensions (e.g., brightness and saturation) but different from separable dimensions (e.g., shape and color). Importantly, results replicated across tasks and could not be explained by effects of confounding magnitudes or non-magnitude image similarity. In other words, perceived number may not be fully abstracted from co-occurring area but, instead, appears to be perceptually interdependent with it.

Relatedly, C&B claim that number is unique compared to other magnitudes – in terms of ratio and the second-order character. Others have similarly argued that number is uniquely salient (Cicchini, Anobile, & Burr, 2016; Ferrigno, Jara-Ettinger, Piantadosi, & Cantlon, 2017). Recent evidence from our lab, however, goes against the uniqueness claim. For example, we found that when perceptual discriminability between number and cumulative area was matched, area biased children's number judgments more than the reverse (Aulet & Lourenco, 2021b) and children sorted visual stimuli according to area, not number (Aulet & Lourenco, 2021c), suggesting greater intrinsic salience for non-numerical magnitude, and consistent with others who have argued against the uniqueness of number (Leibovich, Katzin, Harel, & Henik, 2017; Newcombe, Levine, & Mixs, 2015). Similarly, Testolin, Dolfi, Rochus, and Zorzi (2020) found that the internal encoding of "mature" computational networks, trained to discriminate stimuli according to number, treated total perimeter and convex hull as comparable to number. These effects were even more striking in "young" networks where the internal encoding was primarily driven by convex hull, not number.

C&B also posit that number, unlike other magnitudes, has a second-order character, such that the estimation of number requires stipulating what is being enumerated. For example, among a collection of shoes, number could apply to individual shoes or, alternatively, pairs of shoes. According to C&B, the representation of number is not set unless a relevant unit is specified (Burge, 2010; Frege, 1884). That is, a group of objects has no inherent number absent this stipulation. For the shoe example, the numerical value is n if considering individual shoes, but $n/2$ if considering pairs of shoes.

We agree that, when reasoning in this way, number exhibits a second-order character. C&B, however, apply this logic to the *perception* of number, which we would argue conflates the percept with the concept (Halberda, 2019). They describe "dumbbell" studies in which participants underestimate individual dots that are connected by lines to form dumbbells (e.g., Franconeri, Bemis, & Alvarez, 2009). According to C&B, this effect suggests a second-order character for perceived number because participants' number perception changes in the absence of changes to non-numerical properties (besides connectedness). However, if number perception was genuinely second order, then it should be just as easy to continue perceiving the number of dots, instead of being biased toward the number of dumbbells. But this is not the case! Number percepts are not as flexible as number concepts. Number perception is constrained by physical (e.g., spatial individuation) and Gestalt principles (e.g., common motion; Wynn, Bloom, & Chiang, 2002). Similar constraints (e.g., color grouping) apply to the "ratio" experiments with infants (e.g., McCrink & Wynn, 2007) described by C&B. Moreover, number is perceived in accordance with these principles when arrays are passively, or even unconsciously, viewed (DeWind, Park, Woldorff, & Brannon, 2019; Fornaciai & Park, 2021; Lucero *et al.*, 2020; Van Rinsveld *et al.*, 2020). Accordingly, we suggest that number perception, similar to the perception of other magnitudes, is a first-order property. We can *conceive* and count individual shoes, or the pairs of shoes they make up, but we *perceive* individual shoes. We can *conceive* and count dumbbells, or their individual component dots, but we *perceive* dumbbells.

In summary, although we agree with C&B's description of the ANS as a perceptual system, we would argue that perceived number is not abstract, as it is to a conceptual system with access to symbolic representations such as number words. We have argued that perceived number may not be independent of other magnitudes and it does not appear to exhibit a unique status, including second-order character – calling into question the existence of an ANS that represents rational number.

Conflict of interest. The authors declare no competing interests.

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Sizes, ratios, approximations: On what and how the ANS represents

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Abstract

Clarke and Beck propose that the approximate number system (ANS) represents rational numbers. The evidence cited supports only the view that it represents ratios (and positive integers). Rational numbers are extensive magnitudes (i.e., sizes), whereas ratios are intensities. It is also argued that WHAT a system represents and HOW it does so are not as independent of one another as the authors assume.

Some maintain that we do not have sufficient evidence to establish the existence of the approximate number system (ANS) – that experimental results fail to convince that there is sensitivity (in line with Weber's Law) to the "numerosity" of collections of individuals rather than certain potential confounds such as the total area of the disparate region covered by those individuals. Clarke and Beck (C&B) effectively refute such scepticism. They point to the existence of cross-modal studies, in which, for example, the "numerosity" of a collection of dots is compared to that of a collection of tones, and ask the pointed question – what can the alleged confound be in such a case? They also draw attention to the dumbbell effect, which provides strong evidence that the sensitivity of the ANS is to the discretely varying size of a collection of individuals – a second-order property of a given scenario – rather than a continuously varying magnitude, such as that of the area covered by those individuals. Such results (alongside myriad others) leave no grounds for reasonable doubt that the ANS exists.

But what does this "Number Sense" represent? C&B suggest that the ANS represents, well… numbers – and more specifically, *rational* numbers. They also hope to show that appeal to facts about imprecision in the representational capacity of the ANS does not preclude such an answer, or support an alternative one, such as Burge's (2010) view that the cognitive system in question represents the "pure magnitudes" theorized by Eudoxus in antiquity.

Peacocke (2015) has clarified that Eudoxus' pure magnitudes are extensive, meaning that they can be added to one another: If we take an object with mass $m1$, and combine it with an object with mass $m2$, the result is an "object" with mass $m1 + m2$. Intensities, by contrast, cannot be added. Carey (2009) discusses density, which comes in degrees. We can say how dense something is (in comparison with other things), and even measure this quantitatively. Nevertheless, the density of an "object" that results from combining two objects with densities $d1$ and $d2$ cannot be assumed to be the sum $d1 + d2$ – it depends on the relative sizes of the two objects that are combined! (The reason, of course, is that density is ultimately a relation between two extensities, the mass of an object and its volume.)

C&B argue that the ANS represents rational numbers, and that this suggestion has ecological validity, because it is useful to an organism to represent, for example, probabilities (which are often – although not always – determined by certain ratios). Now, rational numbers are extensive magnitudes: it makes sense to ask how much ½ + ¾ is. But, as far as I can see, C&B cite no evidence that suggests additivity here. Take the (wonderful!) lollipop experiment they discuss: Infants can succeed in choosing a jar with a greater chance that a lollipop randomly selected from it will be of their preferred flavour; yet this only requires that they represent the ratios of their preferred flavour to the other flavour (or to the total). Ratios, however, are intensities: We can compare them; but it makes no sense to ask what 1:2 + 3:4 is. (Indeed, "one is to two plus three is to four" is ungrammatical.) Perhaps, the conclusion that rational numbers are represented (rather than ratios) is premature.

C&B are also keen to distinguish the question of *what* the ANS represents from that of *how* it does so: but care is required in practice to do so. Their view appears to be that the ANS does not represent numbers in the abstract, as objects; rather, it attributes number properties to collections of individuals – in its approximative way. But, if the ANS attributes a numerical size to a collection of objects, we can surely ask what property exactly it represents that collection as having – and it seems we can distinguish the views that it attributes *being (roughly) such and such size* (which is, in fact, numerical, being a size of a collection) and that it attributes *being (roughly) so numerous*.

What would answer the question? Presumably, something about the processing sensitivities of the ANS – although no theorist should embrace the strong sensitivity principle C&B articulate, for the reasons they give. And C&B are surely right that the ANS does not represent magnitudes that are indeterminate in kind between species that vary continuously and species that vary discretely – there are no such magnitudes (even if there are "pure" continuous magnitudes that are, for instance, neither spatial distances nor temporal durations). Yet it might represent numerical sizes without representing them as varying discretely. Arguably, this would be so if the only computations performed on the representations were well-defined on continuously varying magnitudes as well, such as comparison and addition/subtraction. (A system that also exhibited sensitivity to whether there is a one–one correspondence between two collections might be said to represent certain magnitudes as cardinal numbers; and one that displayed a sensitivity to the immediate successor relation might be taken to represent some magnitudes as natural numbers, if these are taken to be things related to zero by the ancestral of that relation.) Is this only a question of *how* the (numerical) magnitudes are represented?

In any case, it seems there is a difference between attributing the properties of *being eight in number* and *being* ***roughly*** *eight in number*: If the collection to which the property is attributed has nine items in it, the second attribution is correct, whereas the first is not. Therefore, this distinction would appear to concern what is represented, not how it is represented. Perhaps, C&B will say this shows instead only that cognitive episodes involving the ANS have accuracy conditions, which admit of degrees, rather than veridicality or truth conditions, which do not – and that it is indeterminate what (i.e., which property) is represented by the ANS?

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Numerical cognition needs more and better distinctions, not fewer

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Abstract

We agree that the approximate number system (ANS) truly represents number. We endorse the authors' conclusions on the arguments from confounds, congruency, and imprecision, although we disagree with many claims along the way. Here, we discuss some complications with the meanings that undergird theories in numerical cognition, and with the language we use to communicate those theories.

We agree that the approximate number system (ANS) represents number and aim to clarify theoretical arguments that are entangled in questions about terminology. What do we mean by area, number, numerosity, and representation?

Although the authors are right that the "argument from congruency" and the "argument from confounds" ultimately fail, some evidence bolstering those arguments is shaky in the first place. When we use *physical* area to infer the relative contributions of continuous and discrete stimulus properties to quantity judgments, we're neglecting a history of psychophysical evidence that perceived and physical area differ (Barth, 2008). Empirical support for this idea came from three experiments in which cumulative area judgments were driven by perceived, not physical, area. Some "arguments from congruency" depend on interpretations based on physical area (e.g., Hurewitz, Gelman, & Schnitzer, 2006; Rousselle & Noël, 2008). Yet quantity judgments can yield apparent congruency effects that disappear when perceived area is considered instead of physical area (Barth, 2008). Incorporating perceived area won't resolve controversies surrounding discrete versus continuous quantity (see Aulet & Lourenco, 2021; Savelkouls & Cordes, 2020; Yousif & Keil, 2020). Nevertheless, to identify processes underlying quantity judgments, *subjective* magnitudes should be explored as potential behavior cues. Otherwise, we'll get the wrong idea about whether number influences area or vice versa, or both, or neither.

We also have to be clear on the terms "number" and "numerosity." We were surprised at the authors' lengthy condemnation of "numerosity." In our usage, "numerosity" refers to a property of a stimulus, not a representation. An array of dots (or string of sounds) has a numerosity. That numerosity is larger when the elements are more numerous. If we've used the phrase "numerosity representation," we weren't referring to woolly "number-like properties" (Burge, 2010). We meant "mental representation of number that refers to the numerosity of a stimulus." It's not a hedge – it's shorthand.

Do other psychologists share our understanding of what "numerosity" means, in which case the target article is simply wrong that our language implies "an assumption that, strictly speaking, the ANS represents numerosities, not numbers" (Clarke & Beck [C&B], sect. 6, para. 4)? Or, are psychologists' uses of "numerosity" inconsistent? We think C&B (and Burge) are wrong about what "numerosity" means to researchers, but either way they've done a service in exposing this confusion, and the field had better get clear about what it does mean.

That said, dropping "numerosity" for "number" isn't the answer. "Number" is ambiguous, and ambiguity breeds confusion. "Number" can refer to number words and Arabic numerals (i.e., symbols for natural numbers) or a property of stimuli (i.e., numerosity) or mathematical entities. For psychologists, it is useful to have a term that unambiguously refers to the number of items in a stimulus. "Numerosity" allows psychologists to discuss discrete quantity without endorsing commitments about how it is represented in the mind.

The target article itself suffers from terminological confusion, over "number" and, at times, "representation." Use of "number" when the authors appear to intend "natural number" frequently obscures their meaning. (We spent considerable time decoding what was meant by each instance of "number"!) And C&B seem to answer claims about what is made explicit by a representational system with arguments about the contents of representations within that system. For example, Carey (2009) argued that the ANS *as a representational system* cannot grant natural number concepts to an organism. But this is not a critique of the idea that ANS representations have true numerical content! Carey (2009) is clear that the ANS represents number: "that analog magnitude representations constitute one system of number representations deployed by human adults has been established beyond any reasonable doubt" (p. 131) and "analog magnitudes are explicit symbols of approximate cardinal values of sets" (p. 135). Carey's argument doesn't attempt "to undermine the hypothesis that the ANS represents number" (C&B, sect. 5, para. 1).

Furthermore, C&B wave away the question of "modes of presentation," arguing that the same property under different modes of presentation is still the same property. Therefore, they argue, the word "number" should suffice to describe that property. For psychologists, however, mode of presentation is not an afterthought. How do different representations of identical aspects of the world map on to each other in the mind? Which modes of presentation subserve word learning, computational tasks, and behavior?

When we ask a question like "where do human number concepts come from," we see that the use of a single word like "number" elides questions of interest. The ANS as a representational system does not encode exactness or the successor function, essential components of natural number. This limitation is important in querying what roles the ANS can play in learning. We concur that ANS representations don't serve as the conceptual source of precise number concepts (Carey & Barner, 2019), and empirical evidence indicates that children don't learn number word meanings via mappings to ANS representations (Carey, Shusterman, Haward, & Distefano, 2017). The fact that the ANS encodes some aspects of number (e.g., its second-order character), but not others (e.g., exactness), highlights the importance of using more specific terminology to clarify which aspects of number, and which properties of relevant representations, are under discussion. The authors' push to use the term "number" promiscuously has a muddying effect rather than a clarifying one.

Eronen and Bringmann (2021) argue that theory development in psychology suffers, in part, from "the relative lack of robust phenomena that impose constraints on possible theories" and "problems of validity of psychological constructs." Numerical cognition is rich in robust phenomena, and construct validity is coming along. But we have an enduring terminology tangle. Carey (2009) wrote: "It then becomes a merely terminological matter whether one wants to use the term 'number' only for natural number or for the integers or for the integers plus the rationals plus the reals (in which case there is no core cognition of number) and adopt some other term for the quantificational content of core cognition systems" (p. 297). Maybe it's not so "merely" after all.

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The number sense does not represent numbers, but cardinality comparisons

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Abstract

Against Clarke and Beck's proposal that the approximate number system (ANS) represents natural and rational numbers, I suggest that the experimental evidence is better accommodated by the (much weaker) thesis that the ANS represents cardinality comparisons. Cardinality comparisons do not stand in arithmetical relations and being able to apply them does not involve basic arithmetical concepts and operations.

Clarke and Beck (C&B) vigorously defend the thesis that the approximate number system (ANS) represents number, which they take to include the natural numbers and the rational numbers (fractions). Although they present compelling responses to some (but not all – see below) objections to their view, the evidence that they present seems consistent with the much weaker thesis that the ANS represents different types of cardinality comparisons. My challenge to C&B is to explain why we need anything more than cardinality comparisons to account for the operation of the ANS.

To explain the simplest form of cardinality comparisons, we can begin with the concept of *equinumerosity*. Informally speaking, two sets are equinumerous when they have the same number of members. In mathematical logic, equinumerosity is standardly understood in terms of there being a 1:1 mapping (a *bijection*) between the two sets. This concept does not, of course, involve any reference to number or numbers, which is why it is the foundation for the influential approach to understanding numbers in the philosophy of mathematics known as *logicism*. But there is no need for fancy mathematical machinery to put this concept to work – simply pairing each apple with exactly one orange and each orange with exactly one apple will establish that a set of apples and a set of oranges are equinumerous.

Much of the experimental evidence cited in support of an ANS takes the form of demonstrated sensitivity to situations where two sets or collections are *not equinumerous*. Therefore, for example, Xu & Spelke (2000) showed that infants habituated to a display with 16 dots dishabituate to displays with a number of dots differing from 16 by a sufficiently large ratio (to a 32-dot array, but not one with 24 dots, e.g.). C&B want to say that this is an example of infants representing (imprecisely) the numbers 32 and 16. But that seems to be using a sledgehammer to crack a nut. Why not simply say that the infants are sensitive to the non-equinumerosity of the two arrays when the ratio between them is sufficiently great? "Equinumerosity" is a fancy word, but a simple concept (think about pairing up the apples and oranges). "Number," in contrast, is a simple word but a (very) fancy concept.

Non-equinumerosity is the simplest form of cardinality comparison, but when non-equinumerosity holds between two sets, one set will always have more members than the other. Representation of the property — *has more members than* – seems to be what is shown by Barth *et al.* (2005) studies of preschoolers, and by any study showing that nonhuman animals reliably select sets with larger numbers of food items.

There are (at least) two good reasons for preferring comparative cardinalities to numbers in explicating the ANS. The first is parsimony. To represent comparative cardinalities is to represent a relational property between two sets. To represent numbers is to represent abstract objects that stand in certain arithmetical relations to each other – representations that can be manipulated according to well understood rules and operations. C&B correctly point out that what they term the strong sensitivity principle is misplaced. It is perfectly possible to represent something without representing *all* of its essential properties. However, because C&B readily concede that there is no evidence that the ANS is sensitive to the successor function or to basic arithmetical operations it seems a good idea to look for representational abilities that are independent of such functions and operations. After all, it does seem impossible to represent something without representing at least *some* of its essential properties, and if one takes away the functions and operations that define the number system, and allows numbers to be represented as imprecise, then no essential properties of the number system are left to be represented.

The second reason has to do with the performance profile of the ANS, which C&B are at pains to emphasize. The ANS appears to conform to Weber's Law. Weber's Law is a law governing discriminability. Typically, it is used to characterize the perception of just-noticeable differences in psychophysics. Such differences are, by their very nature, relational and comparative. Therefore, one would expect the representational currency of any system to be relational and comparative. Comparative cardinalities fit this description better than numbers. What the ANS does is represent comparative cardinalities such as – *is equinumerous to* –, – *has more members than* –, *has fewer members than* –, rather than absolute properties such as – *has (approximately) 16 members* – or – *has (approximately) 32 members* –. By the same token, the auditory system represents properties such as – *is the same volume as* – and – is *louder than* –, rather than absolute properties such as – *has a volume of 55 decibels*.

A final observation. Comparative cardinalities are not numerical magnitudes (or what C&B call "recherché alternatives to numbers"). They are of course related to numerical magnitudes, but that does not mean that they can only be represented by representing numerical magnitudes. A seed-eating bird can represent that one container has more seeds in it than another without representing the first as having 252 and the second as 57, even approximately. By analogy, you or I can represent the sound of a lawn-mower as louder than the sound of distant thunder without representing the first as 90 decibels and the second as 62.

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Positing numerosities may be metaphysically extravagant; positing representation of numerosities is not

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Abstract

Clarke and Beck (C&B) assume that approximate number system (ANS) representations should be assigned referents from our scientific ontology. However, many representations, both in perception and cognition, do not straightforwardly refer to such entities. If we reject C&B's assumption, many possible contents for ANS representations besides number are compatible with the evidence C&B cite.

Clarke and Beck's (C&B's) argument critically relies on the principle that "our search for the referent of a representation should be biased towards entities we have independent reason to posit in our scientifically informed ontology" (sect. 6). This principle is suspect. Many representations cannot be mapped straightforwardly onto entities in our considered, scientific ontology. For good reason: Part of the project of psychology is to understand minds which are unscientific, and whose ontology is mistaken.

Many representations, ranging from previous scientists' beliefs in phlogiston to contemporary Americans' beliefs in paranormal phenomena (Moore, 2005), are not of entities from current scientific ontology. Representations which do have tighter relationships to scientific entities, meanwhile, are frequently confused. Carey (2009) surveys evidence for "undifferentiated representations" both in children and in the history of science: the confusion of heat and temperature (pp. 371–376), and of mass, weight, and density (pp. 379–405). Such mismatches between ordinary representational systems and those of current science are not limited to concepts: It is hotly disputed whether *perceptual* colour, odour, or timbre representations have single, consistent referents from our scientific ontology. If they do, these referents may be relational properties partly defined in terms of the perceiver, or convoluted sets of entities from physics such as wavelengths and chemical compositions, rather than natural kinds. Even perceptual *spatial* representations do not simply map onto Euclidean space, and must be construed as either frequently inaccurate, or as not representing objective, Euclidean spatial properties (Fernandez & Farell, 2009; Hill, 2016; McLaughlin, 2016; Prettyman, 2019). C&B repeatedly accuse numerosity advocates of a double standard, arguing that although number representations are treated as only representing "numerosity," we do not extend this -osity treatment to other entities. But this double standard is a mirage: representations of weight–mass–density, wavelength-osity (commonly known as "colour"), and chemical-composition-osity (odour) have a similarly ambiguous relationship to entities in our scientific ontology.

There are numerous theoretical options for assigning reference to confused or unscientific representations. These include: allowing entities outside our scientific ontology to serve as referents, whether fictional objects, gerrymandered entities such as the property grue, or extra-scientific objects; assigning different scientifically sanctioned entities to the same representation in different contexts; assigning indeterminate referents; or assigning no referents at all. We do not need to choose between these options to see that, given the ubiquity of confused representations, C&B's bias is not a bias we should adopt. This matters: relying too readily on the claim that the approximate number system (ANS) simply "represents numbers" may lead to overconfidence in predicting its behaviour in scenarios where its connection to genuine number is weaker.

C&B's main stated reason for their bias is that it "allows psychological explanations invoking representational content to be integrated with explanations from other sciences, such as biology" (sect. 6, para. 10). However, inter-disciplinary integration is frequently messy, and as a result, similar principles would mislead in similar cases. Consider introducing a bias towards thinking that biological bodies are perfect spheres to allow biology to integrate smoothly with geometry: It is a bias that, if it has any role at all, needs to be extremely weak.

The evidence C&B cite is predicted equally well by views on which the ANS traffics in confused representations, and by the view that it always, unambiguously represents number. To take one example, C&B admit that the ANS is sensitive to many confounds, such as density and size. They point to success on (amongst others) cross-modal number-based tasks, to suggest the ANS represents number rather than density, size, and so on. But, although such behaviour rules out the ANS unambiguously representing one of the potential confounds in all situations, it is consistent with many possible systems which *confuse* number with other confounds. Such a system might be driven by variation in number *in this situation*, especially if other variables it is sensitive to are not available, while ignoring or under-weighting number-specific information in other situations where it produces the very same "number-representations."

How can we empirically distinguish between such possibilities? A full discussion of all potentially relevant forms of evidence is beyond the scope of this commentary. But three potential lines of enquiry stand out. Firstly, investigating details of the ANS' computations: Deciding between some of the possibilities C&B discuss in their account of congruency effects (sect. 3), such as representations of non-numerical variables affecting the inputs, internal processing, or downstream processing of the ANS, would help. Their emphasis on sensitivity to higher order properties also seems promising, but further investigation is called for: *how* does an implicit commitment to the represented variable being higher order play out in the actual computations, and how consistent is this – are there also situations where the ANS is sensitive to first order properties instead, or even confuses higher and lower order properties? Does the ANS consistently respect any *other* distinctive properties of number? Secondly, what is the *degree* to which we find sensitivity to number as opposed to other variables across different conditions? Here, we need to bear in mind that a version of the "file drawer effect" is likely to be particularly pernicious in this case: Results showing *clear* sensitivity to one variable rather than others are more likely to be published. Thirdly, under what conditions do we see failures when the ANS is used in number-based inferences, and can we put any of these failures down to fundamental confusion about number, in a way parallel to results suggesting children confuse weight and density (Carey, 2009, p. 389ff.), or are such confusions extremely hard to come by?

The range of live possibilities for what the ANS represents is vast. C&B's reasons for not taking most of that range seriously rely on a principle which, if applied consistently, would block our understanding of many kinds of perception, conceptual development in children, unscientific adult thought, and even the history of science. We should reject this principle, and with it, anything more than weak confidence in the ANS indeed representing numbers.

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Second-order characteristics don't favor a number-representing ANS

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Abstract

Clarke and Beck argue that the ANS doesn't represent non-numerical magnitudes because of its second-order character. A sensory integration mechanism can explain this character as well, provided the dumbbell studies involve interference from systems that segment by objects such as the Object Tracking System. Although currently equal hypotheses, I point to several ways the two can be distinguished.

Clarke and Beck (C&B) make a convincing argument that the hypothesis that the Approximate Number System (ANS) represents rational numbers shouldn't be rejected. They also argue, in section 5.3, that the competing claim that the ANS represents magnitudes (such as pure magnitudes, Buijsman, 2021; Burge, 2010 or "quanticals," Nuñez, 2017), is less plausible. The crucial point being, according to them, that the ANS is sensitive to a second-order property and the magnitudes, being first-order properties, do not fit the bill. I argue, here, instead that on the view of the ANS as a sensory integration mechanism (Gebuis, Cohen Kadosh, & Gevers, 2016), which readily accompanies magnitude-based views, this second-order character can be explained. The view that the ANS represents magnitudes cannot (yet) be dismissed, as C&B wish to do.

Their first claim is that only a number-based view can explain why the study that found elephants to be sensitive to the number of sunflower seeds in a bucket (Plotnik *et al*., 2019) is insufficient to establish that they have an ANS. In contrast to studies that control for the intensity of the odor, this study fails to show that different sensory modalities are integrated in a single place. Elephants can succeed at the task using a single sensory modality, namely their sense of smell. Hence, the study fails. C&B argue that it does by appealing to the second-order character of rational numbers, whereas this study (may) only measure the first-order property of intensity of smell. They also claim that a magnitude-based view has trouble explaining this, as magnitudes are first-order properties.

I disagree. On the view where the ANS represents pure magnitudes one can easily appeal to the idea of a sensory integration mechanism to account for this phenomenon. Pure magnitudes are ratios of quantities (e.g., 2 cm:3 cm, which can also be obtained by 2 kg:3 kg) and fit in naturally with the idea of a system that integrates the information from different sensory modalities, as they are not specific to any one modality. Therefore, to return to the elephant study, it fails to show that they have an ANS because the task can be solved with specific magnitude representations (for smell) and doesn't require the use of pure magnitudes (which result from an integration across modalities). In this way, it can be just as easily explained why the lack of confounds is problematic for studies that aim to establish the use of an ANS. The sensory integration mechanisms account for this, just as it can answer C&B charge that the argument from confounds is ad hoc. If the ANS is best viewed as a sensory integration mechanism representing pure magnitudes, then we would expect that precisely this mixture of confounding quantities is what drives the responses in different tasks.

The dumbbell studies (Franconeri, Bemis, & Alvarez, 2009; He, Zhang, Zhou, & Chen, 2009) are trickier to explain on the basis of the sensory integration mechanism. C&B already point out that non-numerical confounds are nearly identical whether the dots are connected or not, so that such an appeal is implausible. Yet a different kind of confound should be researched. The stimuli used in both studies have a relatively small number of connected dots/squares, which may readily be picked up by the Object Tracking System (OTS, cf. Feigenson, Dehaene, and Spelke, 2004), which can inform numerical judgments. For example, the fourth experiment of Franconeri et al. (2009) has four circles, and in the connected format these form two dumbbell shapes. Because the connected items are visually far more salient than the remaining dots/squares, it could be that the OTS's processing of these connected items interferes with the target estimation of the four circles. Franconeri et al. (2009) do have a few experiments where the number of connected squares is above the OTS threshold, but in these specific cases there is a significant difference between the connected and non-connected stimuli, re-introducing non-numerical confounds (as Franconeri et al., 2009 already note as motivation for their fourth experiment). He et al. (2009) have more dots, but only ever connect up to two pairs of them, keeping the possible OTS confound. Studies controlling for non-numerical confounds with more connected dots, possibly combined with research on the role of the OTS in these cases, will clarify the situation.

The dumbbell studies, then, might have picked up interference from another (number-related) system that is known to involve object detection, the second-order characteristic that C&B focus on. The remaining second-order aspects are readily explained by a sensory integration mechanism. Yet they also appeal to the second-order character to argue that the number-representing view satisfies their weak sensitivity principle, which states that the ANS should be sensitive to at least some of the essential properties of what it is said to represent. Does the quantical/pure magnitude view similarly satisfy weak sensitivity? As I discuss in Buijsman (2021), all the essential features of quantities, as formalized in measurement theory (ordering, concatenation, and choice of measurement), fit the ANS data. Specifically, its approximate character fits with the formally established flexibility in choice of measurement scale. Therefore, indeed, the sensitivity requirement is satisfied by the magnitude-based view.

If the magnitude-based view can't be ruled out on this basis, what should the next steps be? As discussed, more research on whether systems that involve object identification interfere with the dumbbell studies might give more clarity. In Buijsman and Tirado (2019), we've outlined ways in which this issue might be resolved based on spatial-numerical associations, specifically by studying whether the representations of the ANS are amodal (and shared with symbolic number representations) or modality-specific. Furthermore, all of the future research suggested by C&B can proceed without the need to settle this issue of the representations. Yet, just as they argue that it was too early to rule out the hypothesis that the ANS represents numbers, they were too eager to dismiss the quantity view.

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Numbers in action

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Abstract

To understand the number sense, we need to understand its function. We argue that numerosity estimation is fundamental not only for perception, but also preparation and control of action. We outline experiments that link numerosity estimation with action, pointing to a generalized numerosity system that serves both perception and action preparation.

Clarke and Beck do an impressive job reviewing, and largely refuting, objections for the existence of a number sense, such as congruency, confounds, and imprecision. Arguments of this type are not new. For example, imprecision is at the basis of the well-known philosophical "problem of the speckled hen," presented by Gilbert Ryle to Ayer (1940): "Consider the sense datum yielded by a single glance of a speckled hen: how many speckles does the datum comprise?." That humans cannot enumerate the number of speckles was considered a major challenge to prevailing philosophical theories about "given, direct experiences." Why did Ryle choose number for his challenge, rather than the color, height, or weight of the hen, all equally impossible to judge with great precision? Clearly, our inability to enumerate a discrete number of specks makes the point more intuitively. Perhaps, it is the digital nature of numbers, which implies a discrete and precise description; or perhaps because we have multiple ways of measuring number, including rapid but approximate estimation (approximate number system [ANS]), systematic, and errorless serial counting, as well as exploiting grouping strategies (Anobile, Castaldi, Moscoso, Burr, & Arrighi, 2020a; Starkey & McCandliss, 2014). We can, therefore, internally check our rough numerosity estimation, readily betraying its imprecision: checking analog attributes requires instruments such as photometers, tape-measures, or scales. However, the fact that numerosity can be gaged in various ways, with variable precision, does not refute the existence of a number sense. On the contrary, that number estimation is imprecise and essentially noise-limited is further evidence that it is a sensory system. Ayer did not have the concept of noise-limitation in 1940 (introduced a few years later to psychology and physiology), but correctly anticipated that although the hen does have a definite number of speckles, the sense datum has only an imprecise guess: essentially, the approximate number system.

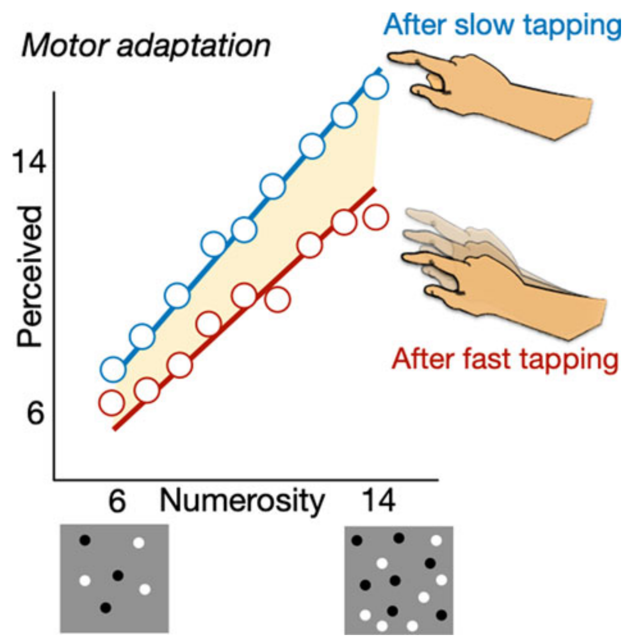

Figure 1 (Burr et al.). Effects of motor adaptation on perceived numerosity. Average perceived numerosity after a few seconds of slow (blue) or fast mid-air tapping (red), as a function of physical numerosity.

Perhaps, the more pressing question is not so much whether a number sense exists, or what class of numbers it encodes, but what purpose does it serve? Has it evolved primarily for estimating the number of speckles on a hen? – or the number of times the hen pecks? – or to help control and monitor the hen's own pecking behavior? It is likely that all are relevant, but the role of perception in action has traditionally been underrated (Goodale, 2014). We argue that numerosity perception is intrinsically linked with action. This is particularly clear in complex tasks such as ballet routines, music production, and the extraordinary waggle dance of bees. But action and number are strongly linked in most movement tasks, such as walking, talking, or eating. It is, therefore, perhaps not surprising that neurons have been identified in monkey cortex that are selective to the number of actions the monkey makes, either turns or pushes (Sawamura, Shima, & Tanji, 2002).

We have used adaptation techniques to reveal a strong link between action and number estimation in humans (Anobile, Arrighi, Togoli, & Burr, 2016). Participants tapped in mid-air with their dominant hand, either very quickly, or around one tap per second. Fast tapping caused robust underestimation of the numerosity of subsequently presented stimuli, and slow tapping caused robust overestimation (see Fig. 1).

The effects were large, around 25%, and equally strong for estimating the number of items in a spatial array as for the number of events in a temporal sequence. This reinforces evidence of a generalized sense of number, spanning space, time, and sensory modality (Arrighi, Togoli, & Burr, 2014), and shows that this general sense is strongly linked to action. Importantly, adaptation (either to tapping or to sequential stimuli) does not generalize over the entire visual field but is confined to the immediate spatial vicinity where the hand had tapped or the stimuli presented (irrespective of the tapping hand). This demonstrates a spatially specific perceptual origin, rather than adaptation or a more general cognitive effect (such as internal counting). Interestingly, the spatial selectivity (for tactile sequences) is as strong in the congenitally blind as in sighted participants (Togoli, Crollen, Arrighi, & Collignon, 2020), showing that visual experience is unnecessary.

The effect of adapting to hand-tapping on perception was not limited to numerosity, but observed also with duration and spatial location estimates (Anobile, Domenici, Togoli, Burr, & Arrighi, 2020b; Petrizzo, Anobile, & Arrighi, 2020). This again is to be expected, given the close links between space, time, and number (Walsh, 2003), and their clear role in action (especially, time).

Other links between action and numerosity perception have been reported with saccadic eye movements. Observers can saccade very quickly toward the more numerous of two arrays, implying a link between action and numerosity systems through dedicated pre-attentive mechanisms (Castaldi, Burr, Turi, & Binda, 2020). At the time of saccades, numerosities of spatial arrays are grossly underestimated, paralleling the effects on temporal duration and spatial extent (Burr, Ross, Binda, & Morrone, 2010). Saccades also affect symbolic numbers: Participants underestimate the results of additions and subtractions when digits are presented at the time of saccades (Binda, Morrone, & Bremmer, 2012). Pupil size is modulated by perceived numerosity, even in the absence of a psychophysical task (Castaldi, Pomè, Cicchini, Burr, & Binda, 2021).

All these results reinforce the existence of an approximate number system in humans, and show that this system encodes numerosity in a generalized manner, across space and time and sensory modality, for use in both perception and action (Anobile, Arrighi, Castaldi, & Burr, 2021). As perception and action are strongly linked in everyday life, the emergence of a sensorimotor mechanism would seem to be a parsimonious and evolutionary useful strategy. For these functions, natural numbers (which include the fascinating case of zero; Nieder, 2016) are sufficient, but we cannot exclude the possibility that the same system encodes rational numbers such as fractions when required.

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Real models: The limits of behavioural evidence for understanding the ANS

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Abstract

Clarke and Beck use behavioural evidence to argue that (1) approximate ratio computations are sufficient for claiming that the approximate number system (ANS) represents the *rationals*, and (2) the ANS does not represent the *reals*. We argue that pure behaviour is a poor litmus test for this problem, and that we should trust the psychophysical models that place ANS representations within the reals.

Clarke and Beck (C&B) ask what the approximate number system (ANS) represents, but an equally important question is what approach should we use to answer this question? C&B put behavioural evidence above all other – arguing that (1) behaviourally attested ratio computations are sufficient for the claim that the ANS represents the rationals, and (2) the absence of evidence that the ANS can compute π or $\sqrt{2}$ shows that it (probably) cannot represent the reals.

However, there are meaningful challenges with both these arguments.

First, C&B rely primarily on research showing that observers can reason about *ratios* of approximate number representations. For example, observers are not only able to judge that the side with 20 dots appears more numerous than the side with 10, but also that one side has (approximately) *twice as many* dots as the other (Jacob, Vallentin, & Nieder, 2012). However, this behavioural finding does not guarantee that ANS representations, *per se*, are rational. It remains entirely possible that whole-numbered ANS representations merely serve as inputs for ratio computations, which are outputted to an entirely separate, domain-general ratio scale (Luce, Steingrimsson, & Narens, 2010; Matthews & Chesney, 2015). Such a scale could represent the ratio of not only two ANS representations, but also two lengths of lines, two sizes of objects, or even – in cross-modal matching tasks – be used to make "a sound three times as loud as [a] light is bright" (de Hevia, Vanderslice, & Spelke, 2012; Ellermeier, Kattner, & Raum, 2021).

C&B are aware of this possibility, but puzzlingly note that it matters little for their arguments, as both the ANS and something like a ratio processing system (RPS) follow Weber's law. Yet adherence to Weber's law does little to unify these competing accounts. Noisy ANS representations of whole numbers could easily show Weber's law in ratio computations: An observer seeing a collection of 20 versus 10 dots could input a noisy signal of 20:10, 18:12, or even 22:9 dots into a ratio operation and outputted to an RPS. This system would then inherit Weber's law without sharing any other properties with the ANS. In fact, despite C&B suggesting that ratio computations are likely encapsulated within the ANS, there is good reason to suspect that these computations can exist entirely separately from it. A domain-general ratio scale easily explains how cross-modal matching tasks are accomplished (e.g., readily matching the ratio between sets of dots to two lines; Bonn & Cantlon, 2017). Moreover, individual differences in the ANS do not correlate with ratio operations in other perceptual domains (e.g., length and area; Dramkin & Odic, 2020; Odic, 2018), suggesting that the ANS is not the bottleneck for ratio computation. But, if ratio computations "live" in an entirely separate system, then C&B have only presented behavioural evidence that *this* system represents the rationals, not the ANS, itself.

A second challenge is in the argument that the ANS does not represent the *reals*. Both C&B, as well as Laurence and Margolis (2005), nicely frame this as the question of the "grain of representations": what is the *minimal unit* on the scale of the ANS? Both sets of authors argue that until we find behavioural evidence that, for example, the mind represents π or $\sqrt{2}$, we should not claim that the ANS is a real-numbered system, harkening to classic debates surrounding perceptual grain sizes. Fechner (1887) famously proposed that the unit of any perceptual dimension is the point at which observers fail to notice an objective increase in that quantity (i.e., the just noticeable difference; JND). But many psychophysicists, including Stevens (1961, 1957), argue that such behavioural "scales of confusability" tell us little about the units of perception because performance factors *always* interfere with true competency.

Consider the case of absolute thresholds of light detection. In ideal situations, even a single photon of light can excite a rod cell in the retina. But observers don't always detect this, likely because of biological noise in the optical nerve or because of a balance that vision has to make between accepting signal versus rejecting noise (Ala-Laurila & Rieke, 2014; Barlow, 1956; Rieke & Baylor, 1998). As a result, pure behavioural signatures are a poor indicator of a perceptual system's true capabilities, which is why researchers studying absolute thresholds use *models* of performance coupled with potential sources of noise in the signal, biology of the eye, *and* the observer's decision making to understand absolute thresholds (Field, Sampath, & Rieke, 2005). In the same way, pure behavioural data are too poor (from performance limitations) to tell us whether observers can represent the infinity of π versus merely 355/113.

What, then, can we conclude about ANS representations if we go beyond pure behaviour? Much like the case of light perception, the best available approach is to rely on models of the ANS, including signal detection approaches that model perceptual signals as highly continuous and in the domain of the reals (Piazza, Izard, Pinel, Le Bihan, & Dehaene, 2004; Pica, Lemer, Izard, & Dehaene, 2004). Models fill the gaps between behavioural data, especially at the limits of behaviour where performance factors occlude true competency. Thus, the onus on proving that the grain size of the ANS is at the level of rationals must fall on demonstrating where current models of the reals break down. In other words, what does the model predict that is explicitly tested and not shown in behaviour (given all appropriate controls, sources of noise, etc.)? Ultimately, without an alternative to challenge the orthodox model of the ANS proper, we can only commit to the reals.

Although we agree with C&B that many nuances of ANS representations have not yet been derived nor tested, determining what the ANS represents will rest on settling what evidence is sufficient. As we've argued, the challenge with relying on behaviour as the sole source of evidence is that it is neither sufficient to tell us that the ANS represents the rationals nor that it doesn't represent the reals. Instead, we should rely on psychophysical models that can go beyond the limits of behaviour to get at the true capabilities of perception and thought.

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The approximate number system represents magnitude *and* precision

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Abstract

Numbers are symbols manipulated in accord with the axioms of arithmetic. They sometimes represent discrete and continuous quantities (e.g., numerosities, durations, rates, distances, directions, and probabilities), but they are often simply names. Brains, including insect brains, represent the rational numbers with a fixed-point data type, consisting of a significand and an exponent, thereby conveying both magnitude and precision.

Despite our title (Gallistel & Gelman, 2000), I agree with Clarke and Beck (C&B) that the approximate number system does not represent irrational numbers. Most of them cannot be represented in any way, because they are unidentifiable (Chaitin, 2005). None can be represented exactly by any physically realized system.

Gelman and I argue that human quantitative reasoning is founded on a prelinguistic system for representing both discrete and continuous quantity, which is phylogenetically and ontogenetically primitive. It provides concepts to which toddlers map the count words as they learn to count, and it supports adult reasoning about quantities. We called the neurobiologically realized symbols in a brain's system for representing quantities *numerons*. Gelman and Gallistel (1978) stressed that numerons both refer and are the object of the arithmetic operations by which brains draw conclusions about the referents.

C&B don't like *numerosity*. Gelman and Gallistel (1978) used *numerosity* to refer to the property of the easily counted sets that are conventionally represented by a number. A *number*, as we understand it, is a player in the game of arithmetic, defined by the rules of that game. In the section of our Chapter 11 headed "The Laws of Arithmetic and the Definition of Number," we quoted the mathematician Knopp as follows: "Every system of objects for which this is true is called a *number system*, because to put the matter baldly, it is customary to call all those objects *numbers* with which one can operate according to the fundamental laws we have listed" (Knopp, 1952, p 5; italics his; "this" refers to any object manipulated in accord with the axioms of arithmetic). Having defined number conceptually, we could not define it referentially, which seems to be the only form of definition recognized by C&B and many others (e.g., Carey and Barner, 2019). A fortiori, we could not define number by the fact that a number sometimes refers to the property of a set that we denoted by *numerosity*.

We defined the numerosity of a set operationally as the number you get when you correctly count it, thereby, explicitly rejecting set-theoretic definitions (Frege, 1884). In my view, this usage is both unproblematic and necessary, because, in experimental work on perception, one needs one word for the percept (e.g., "brightness") and another for the corresponding distal stimulus (e.g., "luminance"). In work on number perception, "number" most gracefully denotes the percept – or, in many contexts, the concept. Therefore, we need another word for the distal stimulus. That word has long been – and likely will continue to be – "numerosity." Why some philosophers think there is something dodgy about this usage is a mystery. If they did psychophysical experiments, they would find it unsatisfactory to say "the number sense represents number" (C&B, sect. 6, para. 1); it's equivalent to saying brightness represents brightness.

A coherent discussion of the psychology of number by a convinced materialist like myself requires vocabulary that makes at least three distinctions: (1) number qua arithmetically defined concept; (2) number qua property of a finite set; and (3) number as a symbol in a computing machine like the brain. A number symbol in a computing machine sometimes refers to the property of a set measured by counting it. More often, however, it refers to a continuous quantity, such as a duration. And, perhaps even more often, it is just the name the machine uses for something, for example, the ASCII names for the symbols on a keyboard. Gelman and Gallistel have been tolerably consistent in denoting (1) by *number*, (2) by *numerosity*, and (3) by *numeron*. They did not, for example, title their book, The Child's Understanding of Numerosity.

In 2000, Gelman and I suggested that numerons were noisy magnitudes. We subsequently disavowed that hypothesis (Leslie, Gelman, & Gallistel, 2008). If the numerons that represent distance traveled in animal navigation had 10% noise, path integration would be impossible (see Fig. 1 in Gallistel, 2017). Path integration is well developed even in ants. They count their 13 mm steps over distances of at least 1,300 m (Buehlmann, Graham, Hansson, & Knaden, 2014; Wittlinger, Wehner, & Wolf, 2006), a count that rises to 100,000.

The symbols in physically realized systems for representing quantities and manipulating them arithmetically make only approximate reference to the computable numbers. When efficiency, speed and low energy consumption are strong considerations, engineered number symbols are a fixed-point data type. They have two parts, the exponential part, which specifies the scale, and the significand, which specifies the number of subdivisions distinguished at any given scale. If, for example, 3 binary digits constitute the significands, then a fixed-point binary symbol system distinguishes $2^3 = 8$ different magnitudes at any scale. The number of bits in the exponent specifies the scale. Thus, for example, 0e0 denotes $0 \times 2^0 = 0$; 1e0 denotes $1 \times 2^0 = 1$; and 101e11 denotes $5 \times 2^3 =$ decimal 40. (See Gallistel, 2017 for details, including the explanation of why this system may represent any signed integer – *and* rational numbers that approximately represent quantities such as rates and probabilities.)

The numbers of binary digits in the significands of numerons may be estimated by the reciprocals of the Weber fractions. Weber fractions, generally, fall in the range from 0.0625 to 0.25, which implies 2–5 binary digits in the significands of most numerons.

The small number of binary digits in numeron significands bespeaks the sophistication of basic brain mechanisms: Numerons convey into computations the limited precision with which a brain's measurement operations generate the symbols that carry forward in time information about empirical quantities. These measurement operations, which Gelman (1972) called *estimators*, rarely deliver a precision better than ±10%, whether the quantity measured is discrete or continuous (Cheyette & Piantadosi, 2020; Cordes, Gelman, Gallistel, & Whalen, 2001; Durgin, Akagi, Gallistel, & Haiken, 2009; Gallistel, 2017; Gibbon, Malapani, Dale, & Gallistel, 1997; Halberda, 2016). Representing empirical quantities with more bits in the significands would imply a misleading precision. That can be disastrous, as any navigator should know.

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Non-symbolic and symbolic number and the approximate number system

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Abstract

The distinction between non-symbolic and symbolic number is poorly addressed by the authors despite being relevant in numerical cognition, and even more important in light of the proposal that the approximate number system (ANS) represents rational numbers. Although evidence on non-symbolic number and ratios fits with ANS representations, the case for symbolic number and rational numbers is still open.

The authors (hereafter, C&B) make an interesting argument for the consideration that the approximate number system (ANS) represents number rather than numerosity. The clarity in disentangling *what* is represented versus *how* it is represented is also a valuable contribution to the field. However, there are some issues that the argument leaves unresolved despite being critical to the nature of the ANS. A relevant one is the distinction between the representation and processing of non-symbolic and symbolic number. This is a well-established distinction in the numerical cognition literature, whose dismissal – for example, by reducing it to a matter of *how* number is represented – would be overlooking its depth and implications. Characterizing the ANS as a "primitive and prelinguistic capacity" shared across many species is a clear signal that C&B focus their case on the processing of non-symbolic number. However, the evidence cited mixes data from non-symbolic and symbolic studies (e.g., Henik & Tzelgov, 1982), blurring an otherwise clear and engaging argument. Although C&B distinguish in their exposition between ANS representations and precise number concepts, this distinction is intended to separate ANS representations from the advanced number constructions studied by mathematics, and it does not address the non-symbolic/symbolic contrast.

Myer and Landauer's (1967) study on single-digit number comparison may be considered an essential piece of evidence suggesting that number symbols are to some extent represented by the ANS. These authors showed that young adults' error rates and response times in comparing two digits decrease with increasing numerical distance between them. However, the consideration of these two types of numerical processing, non-symbolic and symbolic, brings an extra layer of complexity to number processing. For instance, symbolic number processing introduces unit-decade-compatibility effects which are meaningless in non-symbolic processing. Nuerk, Weger, and Willmes (2001) presented this concept and proved that the comparison of multi-digit numbers is strongly affected by a competition between the numbers and the digits that compose them: Comparing 42 and 57 is easier than comparing 47 and 62 because in the former case the larger number coincides with the number with the larger decade and unit, whereas in the latter case the larger number has the larger decade but the smaller unit. Effects such as this one are specific to symbolic number representations, as the comparison of a set of 42 blue dots versus one of 57 yellow dots will likely lead to a similar outcome to that of a set of 47 blue dots versus one of 62 yellow dots (both comparisons engage the ANS and the yellow: blue ratios in each case are about 1.3). Altogether, the ANS seems able to represent number symbols, but this representation would be limited to single-digit numbers (see also Nuerk, Moeller, Klein, Willmes, & Fischer, 2011).

When it comes to rational numbers, the distinction between non-symbolic and symbolic processing becomes even more complex, and the research scarcer. Non-symbolic processing in this case refers to the capacity of perceiving and using ratios, whereas symbolic processing brings to the table fractions and decimals. Fractions are visually depicted as two natural numbers separated by a line. There is plenty of evidence that human adults perceive the magnitudes of natural numbers in an automatic manner (i.e., even if it is not relevant for the task, see Henik & Tzelgov, 1982). The magnitudes of fractions, however, seem to be activated only when they are relevant for the task (Bonato, Fabbri, Umiltà, & Zorzi, 2007; Gabriel, Szucs, & Content, 2013; Kallai & Tzelgov, 2012). Children's intuitive reasoning with fractions show important congruency effects (erroneously judging a fraction as larger than another if its components are larger, e.g., concluding that $2/3 < 4/9$ because $2 < 3$ and $4 < 9$; see e.g., Gómez & Dartnell, 2019; Ni & Zhou, 2005; Van Hoof, Lijnen, Verschaffel, & Van Dooren, 2013). It is difficult to ascribe fraction comparison performance to the ANS, however. Although some studies have reported distance effects in response times in fraction comparison tasks, these times are too large to license conclusions about the mental representations of fractions (e.g., Schneider & Siegler, 2010) or the task is to simple to actually engage fraction representations (e.g., Bonato et al., 2007). Nonetheless, adults who are highly mathematically competent also show congruency effects in their response times but also distance effects (Morales, Dartnell, & Gómez, 2020; Obersteiner, Van Dooren, Van Hoof, & Verschaffel, 2013), showing that the discussion is far from over (see also Binzak & Hubbard, 2020, for positive evidence for ANS involvement in fraction comparison).

Ratios (or non-symbolic rational numbers) as a unifying percept of the number sense would be a very compelling theory (e.g., with natural numbers represented as ratios with respect to one). It would be consistent with the measuring function of numbers, common to both natural and rational numbers. As C&B note, natural number counting is not essentially tied to one as the unit, as counting can occur by pairs, tens, or dozens. But, even in this scenario, ratios are a limited aspect of rational numbers and, similarly, non-symbolic numbers are a limited aspect of natural numbers. Although C&B are convincing about the ANS representing non-symbolic numbers and ratios, the case about symbolic ones is less successful. In this regard, it is worth asking to what extent we can restrict our concept of number in order to call numbers to ANS representations. I suggest that the non-symbolic/symbolic distinction is, in this sense, a key one. If the ANS is not convincingly involved in processing of symbolic numbers (naturals and rationals), it would be more parsimonious to claim that it represents ratios rather than rational numbers.

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Representation of pure magnitudes in ANS

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Abstract

According to Clarke and Beck (C&B), the approximate number system (ANS) represents numbers. We argue that the ANS represents pure magnitudes. Considerations of explanatory economy favor the pure magnitudes hypothesis. The considerations C&B direct against the pure magnitudes hypothesis do not have force.

Clarke and Beck (C&B) reject Burge's (2010) hypothesis that the approximate number system (ANS) represents Eudoxian pure magnitudes. They maintain that ANS represents natural and rational numbers. But there are strong reasons – under-appreciated in the target article – to favor pure magnitudes as what ANS represents. C&B's considerations *against* the pure magnitudes hypothesis have no force.

Pure magnitudes can measure both continuous magnitudes (e.g., an object's weight) and the magnitude of an aggregate's membership. (An aggregate is a concrete analog of a set; Burge, 1977.) Pure magnitudes always apply only relative to some property that delimits what is measured. In the continuous case, pure magnitudes apply to something (e.g., some water) relative to a dimension (like weight). In measuring an aggregate's membership, pure magnitudes are like natural numbers in having to apply relative to the sortal that determines an aggregate's members. (The same physical stuff can constitute 1 deck, 4 suits, or 52 cards; Frege, 1884.) Crucially, pure magnitudes occur in ratios. Pure magnitudes are associative and commutative under analogs of addition and multiplication; and, for any magnitudes a and b, exactly one of these conditions holds: $a = b + c$, for some magnitude c; $b = a + c$, for some magnitude c; or $a = b$ (Scott, 1963). They do not, however, bear successor relations: there is no "next" pure magnitude.

The fundamental advantage of the pure magnitudes hypothesis is its comparative explanatory economy. First and most importantly, pure magnitudes have all the structure necessary to explain extant evidence relevant to ANS. Ratio-dependent discrimination behavior comprises the core data in ANS research. As mentioned, pure magnitudes occur in ratios. Number hypotheses concerning ANS representations predict capacities that the data do not support. For example, there is no evidence that ANS capacities include counting, one-to-one matching, or a successor operation. These are basic to a competence in representing natural numbers. They are no part of a competence for representing pure magnitudes. Second, on our view, pure magnitudes are already represented in perception: Continuous magnitudes are measured there by pure magnitudes relative to a dimension (weight and distance). Measuring continuous magnitudes by number would require both a dimension and a unit of measurement. Perception appears to be unit-free (Burge, 2021; Peacocke, 1986, 2019). In sum, the pure magnitudes hypothesis is supported by the evidence, does not posit more than is necessary, and accords well with explanations of perceptual magnitude representation.

C&B, citing Burge (1982), correctly note that some representational competencies do not require sensitivity to every essential feature of what they represent. They also, citing Burge (2005), caution against confusing what is represented with how it is represented (the mode of presentation). Accordingly, one might think that ANS can represent natural numbers despite the absence of counting, one-to-one matching, or a successor operation, and thus that the numbers hypothesis need not postulate these capacities. We deny this. In typical cases, representational competence despite limited sensitivity to essential features is grounded in causal connections to the subject matter or in reliance on interlocutors. These factors are irrelevant here. The main evidence for competence in representing numbers for example, in developmental studies is evidence of capacities for counting, one-to-one matching, and a successor operation. These capacities constitute our main grip on whether numbers are represented.

Why, then, do C&B reject the pure magnitudes hypothesis? They offer two main considerations.

The first invokes the sortal-dependence of membership estimation stressed by Burge (2010). ANS relies on a sortal's distinguishing and grouping the members of an aggregate. Natural numbers *must* measure a magnitude relative to a sortal. A pure magnitude, by contrast, does not *have to* measure a magnitude relative to a sortal (as when it measures weight). C&B claim that pure magnitudes "are thus poorly suited to capturing the contents of ANS representations." This argument has no weight. When pure magnitudes measure aggregate membership, they *must* hold relative to a sortal. That pure magnitudes can also measure continuous magnitudes without a sortal is irrelevant. ANS representations of pure magnitudes can thus be sensitive to the sortal-dependence of the magnitude of aggregates' membership, just as attributions of number would be.

C&B's second consideration is that we should favor the hypothesis that postulates representations of "entities we have independent reason to posit in our scientifically informed ontology." This consideration is *prima facie* (agents can represent there to be entities that do not exist) and is overwhelmed by considerations of explanatory economy of the kind we advance. Furthermore, here the consideration does not favor the numbers hypothesis. C&B claim that scientific explanations refer to numbers, not to numerosities – and so, presumably, not to numerosities construed as pure magnitudes. But, in mathematical science, pure magnitudes are in as good-standing as numbers (Scott, 1963). And, as noted, empirical science is already committed to attributions of pure magnitudes in its explanation of perception. C&B enlist Burge's (2010) use of ethology to settle whether frog vision represents flies or undetached fly parts. The case seems disanalogous. Ethology can break a tie between causal candidates. Different considerations are needed for mathematical entities, which apply to concrete particulars with causal powers only via further properties. Those considerations favor the pure magnitudes hypothesis.

C&B characterize pure magnitudes as "exotic" and numerosities more generally as "recherché" and "peculiar." Quanticals are deemed "mysterious." These formulations could suggest that positing "*ersatz*" numbers is problematic because they are unfamiliar and ill-understood. However, pure magnitudes have been theoretically well-understood since the ancient Greeks. Through their presentation in Euclid's *Elements*, they were central to mathematical and scientific practice up through the early modern period (Stein, 1990; Sutherland, 2006). They are indeed unfamiliar to, and not reflectively understood by, many possessors of ANS, which after all include organisms that may well lack supra-perceptual powers. But numbers are similarly unfamiliar to such creatures. Theorists need not be deterred.

The pure magnitudes hypothesis explains the behavioral data without invoking unevinced capacities (as the neologism "numerosity" cautioned against) and cites resources already deployed in perception. Numbers are more familiar to us. ANS represents pure magnitudes.

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Not so rational: A more natural way to understand the ANS

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Abstract

In contrast to Clarke and Beck's claim that that the approximate number system (ANS) represents rational numbers, we argue for a more modest alternative: The ANS represents natural numbers, and there are separate, non-numeric processes that can be used to represent ratios across a wide range of domains, including natural numbers.

Clarke and Beck (C&B) argue that the approximate number system (ANS) represents rational numbers. This claim is based on a growing body of evidence that suggests humans are capable of comparing ratios between natural numbers. They argue that the most straightforward explanation for this ability is that the ANS represents rational numbers, which can capture ratios between numbers and can themselves be compared. We find C&B's careful argument that the ANS represents numbers, rather than non-numerical confounds, persuasive. However, we find their argument that the ANS represents rational numbers to be less careful, and less persuasive. We argue for a more modest alternative: The ANS represents natural numbers, and there are separate, non-numeric processes that can be used to represent ratios across a wide range of domains, including natural numbers.

The primary argument that the ANS represents rational numbers, as far as we can tell, is that people can represent ratios between natural numbers, and that to do this, they must be using rational number representations. This ratio processing system (RPS), they argue, is best understood as a component of the ANS, and thus the ANS must be representing rational numbers. Surprisingly, the central evidence for this connection between the RPS and ANS is that both systems seem to be governed by Weber's Law, and this similarity in performance is taken to be "suggestive of a shared system." But, as C&B themselves point out, a great number of representational systems, including those for distance, duration, and weight, also seem to be governed by Weber's Law. Yet they specifically need to resist the claim that systems for representing duration, weight, or distance are part of a shared system that also includes number representation. They can't have it both ways, and so even C&B should not think that simply conforming to Weber's Law is sufficient evidence for being part of a shared system.

However, you might wonder whether C&B could provide some other kind of evidence that RPS and ANS form part of a shared system, perhaps evidence that they share some shared neural substrate. But C&B make it explicit that their conjecture concerns only a computational level of analysis, so even if some implementation-level link were discovered, it would not help their argument. They need to rely on functional properties of these systems, but the only one they have to offer is conforming to Weber's Law. Yet that property is shared so widely that it is not much evidence one way or another.

Finally, one might wonder whether C&B may actually be appealing to some more general principle in their argument that the RPS represents rational numbers. Perhaps, they are appealing to the argument that because the RPS clearly represents relationships between numbers, RPS representations themselves must be number representations. But this is not a good form of inference. It is not generally true that a representation of the relationship between two entities is of the same kind as the representation of those two entities. One might have an intuitive sense that a chair and couch are more similar to each other than a chair and a lamp. But clearly, the representation of the similarity between two items of furniture is not itself a furniture representation. Analogously, a representation of the relationship between numbers (such as a ratio) need not itself be a numerical representation, and *a fortiori*, need not be a rational number representation.

Given that this is not a generally valid form of inference, C&B must provide some other form of evidence that ratio representations are numeric. This conclusion is not obvious; for example, we are able to represent the ratios between lengths of lines according to Weber's Law, but these representations don't strike one as necessarily numeric.

The remaining question, then, is whether there is a better way to explain the representation of ratios between natural numbers. Here is one such alternative: The ability to represent ratios between natural numbers in accordance with Weber's law arises from the same general non-numeric ability that allows us to represent relationships between all kinds of things, whether it be a matter of length, weight, duration, color, or whatever else. When we discriminate between different numeric ratios, we may simply be applying this quite general ability to the genuinely numeric representations of the ANS.

And so, C&B have not proven that representations of ratios between natural numbers are necessarily rational number representations, nor have they provided strong evidence that the RPS is a component system of the ANS. A more natural way to understand the ANS is that it simply represents natural, not rational, numbers, and that ratio representations rely on a separate, domain-general process.

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Numerical cognition: Unitary or diversified system(s)?

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Abstract

Many researchers, including Clarke and Beck, describe the human numerical system as unitary. We offer an alternative view – the coexistence of several systems; namely, multiple systems (general magnitude, parallel individuation, and symbolic) existing in parallel, ready to be activated depending on the task/need. Based on this alternative view, we present an account for the representation of rational numbers.

Clarke and Beck (C&B) describe the human numerical system as unitary. They review effects of various perceptual properties (e.g., size or density) on judgments of numerosity in section 3: "congruency" and section 4: "confounds." However, C&B argue that these effects are inconsequential because of the unitary nature of the numerical system. We disagree. We suggest that converging behavioral and neuroimaging evidence has shown that the number system is not unitary but diversified. First, behavioral evidence has consistently reported involvement and influence of basic continuous properties on the processing of numbers (for a review, see Leibovich, Katzin, Harel, & Henik, 2017). Second, neuroimaging evidence has highlighted distinct brain areas associated with various numerical and quantification tasks. Taken together, this evidence obligates re-examination of the premise that the number system is unitary. In turn, this provides interesting insights into the processing of rational numbers.

Throughout this review, the authors draw an analogy between the approximate number system (ANS) and the visual system. They state, "the visual system is often viewed as unified by its function, despite comprising relatively autonomous sub-modules performing dedicated tasks at various levels of visual analysis." For example, the visual system is characterized by two co-existing visual pathways: (1) the primary, evolutionarily younger geniculostriate system, and (2) an evolutionarily older retino-tectal system. Although these two systems have somewhat different roles, they are connected and have mutual effects on vision (Henik, Rafal, & Rhodes, 1994). The relatively less known evolutionarily older system deals mainly with spatial aspects of vision and connects to the parietal lobes, including the intraparietal sulcus (IPS), a key brain region linked to number processing.

The IPS and adjacent brain structures are involved in basic number processing, but also action (i.e., reaching and grasping). Walsh (2003) suggested that important computational demands of an action system (reaching and grasping) are the basis for the involvement of the parietal lobes in comparative judgment tasks. Namely, the activity of the parietal lobe reflects computational demands of the brain dorsal system (that starts at the visual areas of the occipital lobe and connects to the parietal lobe) involved in perception for action (Goodale, 2000). However, it might be the other way around. Specifically, routines and brain structures underlying comparative judgments that are needed for action might have evolved from a single system that originally supported computing magnitudes (e.g., size). In line with this notion, for the dorsal brain system to develop, it was evolutionarily critical to first be able to compute amount or size and size differences. A neurocognitive system that handles this aspect of cognition (i.e., the evaluation of size or amount) might have been foundational for the development of the occipito-parietal dorsal brain system (i.e., the system that supports perception for action). Critically, this same system (i.e., the evaluation of size or amount) was also foundational for the development or advancement of the numerical system. Accordingly, we have suggested the coexistence of two systems (Henik, Gliksman, Kallai, & Leibovich, 2017); an older system that underpins the evaluation of size or amounts of substance and a number system that is discrete in nature and supports the evaluation of precise numerical quantities (Leibovich, Ashkenazi, Rubinsten, & Henik, 2013).

Recent meta-analyses of neuroimaging studies that evaluate the neural correlates of number processing across formats and non-numerical magnitude processing support this idea. Specifically, Sokolowski, Fias, Ononye, and Ansari (2017) show the set of brain regions supporting symbolic and non-symbolic number processing highly overlap with the brain regions supporting non-numerical magnitude processing (e.g., size, length, and luminance). However, symbolic and non-symbolic number processing are also associated with additional, format-specific regions lateralized within the parietal lobes (with symbolic on the left and non-symbolic on the right). This meta-analytic data go against the idea that a single system supports all of numerical cognition, instead suggesting that numerical cognition is supported by diversified systems, one of which is a general magnitude system. Most of the empirical studies included in the meta-analysis use active tasks that involve decision making and motor response (i.e., perception and action), which are known to be associated with the IPS. A recent functional magnetic resonance imaging (fMRI) adaptation study (https://psyarxiv.com/xw2fq/) highlights that symbols, quantities, and physical size have overlapping but also distinct brain regions in the parietal lobes, and quantities and size are quite similar in terms of the patterns of activation whereas symbols are distinct. This reveals overlapping and distinct brain regions supporting numerical and non-numerical magnitude processing in the absence of active tasks. Other recent data from Zimmermann (2018) provide direct evidence that different mechanisms account for the perception of visual numerosity. Specifically, Zimmermann shows that low numbers are sensed directly as a primary visual attribute, but the estimation of high numbers depends on the area/size over which the objects are spread. Hence, subsystems within the two systems proposed above may support computations of particular quantities and amounts.

The proposal that numerical cognition is supported by diversified systems sheds new light on the authors' discussion of rational numbers. C&B conceptualize rational numbers as a representation of numerical ratios among positive integers. They suggest that the ANS first represents natural numbers of concrete pluralities and only then derives ratios therefrom. Within the framework of a diversified numerical system, the magnitude and number systems may operate in parallel to extract the necessary information. Within such a composite system, proportions could be extracted in the way suggested by the authors (based on the number system), or more directly by the magnitude system, or in an orchestrated operation of the two systems.

In summary, we posit the idea that numerical cognition is supported by diversified systems, rather than a unified system. Such a divergent system aligns more closely to the structure of the visual system, is better supported by empirical data in the field of numerical cognition, and provides a more adequate explanation for the way the human mind processes rational numbers.

Conflict of interest. There is no conflict of interest.

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Ratio-based perceptual foundations for rational numbers, and perhaps whole numbers, too?

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Abstract

Clarke and Beck suggest that the ratio processing system (RPS) may be a component of the approximate number system (ANS), which they suggest represents rational numbers. We argue that available evidence is inconsistent with their account and advocate for a two-systems view. This implies that there may be many access points for numerical cognition – and that privileging the ANS may be a mistake.

We applaud Clarke and Beck's (C&B's) compelling use of analogies with other domains of perception to defend the notion that the approximate number system (ANS) truly represents numbers. We were also gratified to see that they arrived at a similar conclusion to that proposed by our own study, one "closely related to a suggestion from the developmental and educational psychology literatures according to which there is a 'ratio processing system' (RPS)." We have previously advanced the view that humans and other animals possess a perceptual system for representing ratio magnitudes, and that they therefore represent rational numbers, rather than being limited to representing purely integers (e.g., Lewis, Matthews, & Hubbard, 2015; Matthews, Lewis, & Hubbard, 2016; see also Jacob, Vallentin, & Nieder, 2012). C&B go on to suggest that it isn't always clear whether the RPS is a separate system from the ANS or a component of it, deciding in favor of the latter. They further argue that "the hypothesis that the RPS represents rational numbers is not always clearly distinguished from the conjecture that it represents real numbers more generally" (sect. 7.3, para. 2). In this commentary, we focus on these points in light of the current empirical record.

Our view is informed by our prior findings that the RPS is operative in multiple visual formats – extending beyond the discrete dot arrays that have typically been the focus of ANS research. We showed that children and adults can also compare ratios made of lines, circles, and irregular blobs (e.g., Binzak et al., submitted; Park, Viegut, & Matthews, 2020; see also Bonn & Cantlon, 2017). Because ratio perception has been demonstrated using various continuous stimuli not typically considered the province of the ANS, we argue (1) that the RPS cannot be a component of the ANS, and (2) that perceiving numerical ratios may be every bit as fundamental as perceiving exact number (or numerosities). A corollary to this position is that one route to whole number representations might be an emergent property of ratio perception (i.e., when the denominator is 1).

Although these issues must ultimately be settled empirically, in the spirit of C&B, we think an analogy from brightness perception illustrates the plausibility of our argument that the ANS and RPS are two systems. Although individual photoreceptors signal absolute light levels, much of the perceptual system is tuned to relative (ratio) brightnesses of different portions of surfaces, such as when identifying edges in a scene or perceiving shades of gray in black and white images. This system yields the same percept even under a 1,000-fold difference in absolute light levels, such as when moving from indoors to outside under bright sunlight. That is, the visual system computes relative brightness as its primary perceptual feature (for a review, see Gilchrist, 2013) and either normalizes or discounts absolute illumination. In parallel, intrinsically photosensitive retinal ganglion cells signal absolute illumination and feed into systems that regulate the pupillary reflex and circadian rhythms (e.g., Yamakawa, Tsujimura, & Okajima, 2019). By analogy, the RPS could be specialized for perception of relative quantity (numerosity), whereas the ANS is specialized for perception of absolute numerosity. Furthermore, as with brightness perception, absolute number may be calculated less frequently and relative number perception may be the predominant mode of perception.

In line with the two-systems view, findings from our labs further suggest that the RPS is not a component of the ANS. For instance, in prior studies, we showed that the predictive power of the RPS was independent of ANS acuity, which contributed almost no explanatory power to the models (Matthews et al., 2016; Park & Matthews, in press). Moreover, if the ANS and

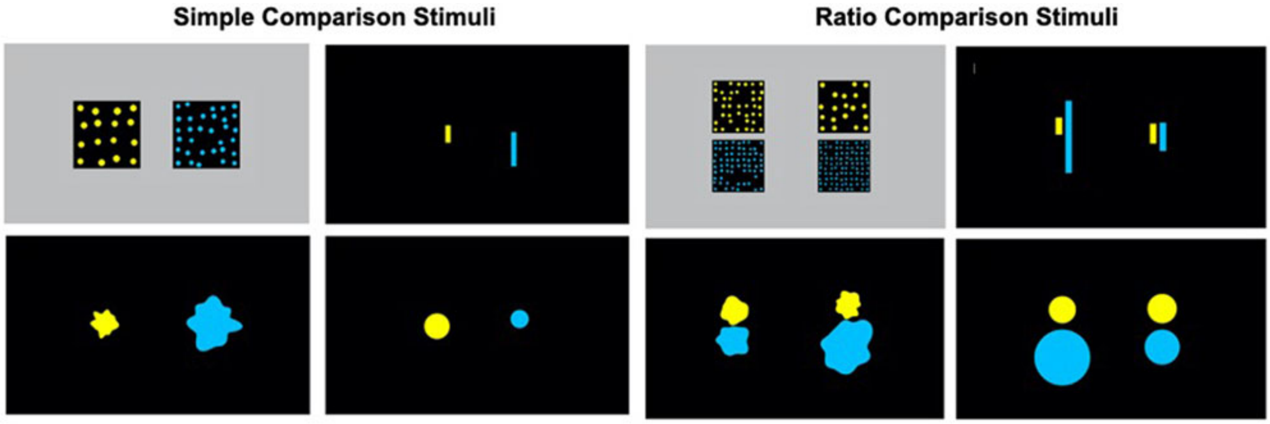

Figure 1 (Hubbard and Matthews). Comparison stimuli used by Park et al. (2020) organized by task type (simple vs. ratio comparison) and by format (dots, lines, blob, and circles).

RPS constitute a single system, we would predict that long-term ANS training – which successfully transferred across visual field locations – should also transfer across tasks to the RPS, but this was not observed (Cochrane, Cui, Hubbard, & Green, 2019). Moving forward, it would be interesting to test whether RPS training transfers to ANS tasks. Most importantly, Park et al. (2020) carried out a battery of tasks using different stimulus formats (e.g., circles, dots, and lines) where participants compared simple stimuli (ANS style) and ratio stimuli (RPS style). They showed that performance was driven more by task similarity (ANS vs. RPS) than by stimulus format (circles, dots, and lines) (Fig. 1).

Despite our preferred take, however, this clearly remains an open question. For instance, in a recent computational modeling study, we trained a deep convolutional neural network (DCNN) to compare non-symbolic numbers, either as simple dot arrays or as ratios composed of two-dot arrays (Chuang, Hubbard, & Austerweil, 2020). Analysis of the hidden unit responses suggested that RPS representations might emerge from tuned (ANS style) units.

More research is necessary for the final adjudication. That said, C&B have done the entire field a service by highlighting that the ANS might be only one component of a multifaceted number sense that integrates various cues and generates various usable outputs from those cues. In highlighting the importance of ratios, C&B underscore that Weber-guided systems can compute not only integers, but also rational numbers. This implies that there may be many access points for numerical cognition – and that privileging the ANS may be a mistake.

As for what type of numbers might be represented by a perceptual number sense, we concur with C&B that the type of number represented may be limited by the nature and precision of the inputs of the perceptual system. The RPS can presumably represent the entire set of x/y for all x and y which a given input system can represent. Thus, if the RPS is truly limited to discrete inputs, then the number sense would include only the rationals. However, if it is more continuous in character, then it could include the reals.

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What are we doing when we perceive numbers?

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Abstract

Clarke and Beck rightly contend that the number sense allows us to directly perceive number. However, they unnecessarily assume a representationalist approach and incur a heavy theoretical cost by invoking "modes of presentation." We suggest that the relevant evidence is better explained by adopting a radical enactivist approach that avoids characterizing the approximate number system (ANS) as a system for *representing* number.

Clarke and Beck (C&B) argue convincingly that the approximate number system (ANS) plays a vital role in allowing us to directly perceive *number* rather than *numerosity*, and we concur (Jones, 2016, 2018). However, by adopting a representationalist stance and appealing to the notion of "modes of presentation," they unnecessarily incur a heavy theoretical cost. By eschewing representationalism and the idea that the *function* of the ANS is to *represent* number, and instead adopting a radical enactivist stance (Hutto & Myin, 2012, 2017; Zahidi, 2021; Zahidi & Myin, 2016), one can explain our direct perception of number with less philosophical baggage.

The issue of whether we have perceptual access to numerical properties is not new, and, over the last century, was largely seen by philosophers of mathematics to have been settled in the negative because of Frege's (1950, pp. 9–14, 27–32) infamous critique of Mill's empiricism, whereby the number we assign to a collection depends on our conceptualizations. As such, there is seemingly no room for number to be a mind-independent property that we directly perceive. This consensus has, more recently, begun to shift, in light of evidence that we have a natural capacity for directly apprehending numerical properties that is perceptual in nature (see Anobile, Cicchini, & Burr, 2016 for a review; Jones, 2018, pp. 150–152).

C&B convincingly argue that we perceive number directly and that the imprecision of the ANS is no reason to suggest that we perceive something other than numerical properties, for example, numerosities. However, this still leaves open the question of what we're perceiving when we perceive number. C&B's answer is that we perceive natural numbers as properties of collections (p. 10) and rational numbers as properties of ratios, yet this is unsatisfyingly trivial, because the terms "collection" and "ratio" merely refer to that to which (natural and rational) numerical properties can apply. Thus, their solution simply raises the question of how we perceive collections or ratios.

C&B take the idea that the ANS *represents* numbers to be the best explanation of the available evidence, but they neglect alternative non-representationalist explanations that accept our sensitivity to numerical properties without committing to any neural system representing those properties using particular modes of presentation. For example, it is possible to understand perception of numerical properties of collections as perception of affordances for engaging in various activities (Gibson, 1979; Jones, 2018; Kitcher, 1984, pp. 11–12, 108). To "perceive a collection of apples as being seven in number" (p. 10) is to be sensitive to structural properties that are significant for a range of actions. The "sevenness" is not a property of the apples, nor of the perceiver, but of what the perceiver can do with them. Rather than the ANS functioning to "keep count of whole items" (p. 34, emphasis removed), it plays a role in enabling actions such as counting. This approach is more closely aligned with recent evidence suggesting that the ANS is a *sensorimotor system*, rather than a simple number detector, because it is implicated in both numerical perception and numerical action, as well as interaction between the two (see Anobile, Arrighi, Castaldi, & Burr, 2021 for a review). This suggests that "the neuronal populations in the theory do not serve as representations of quantity, but serve as causal mediators between input and behavior" (Zahidi, 2021, p. S537).

In making their case that the ANS represents, C&B rely heavily on a familiar philosophical conceit – the idea that represented items appear in specific guises or "modes of presentation." This assumption puts them in position to explain how the ANS can be imprecise despite representing specific numbers. The notion of a mode of presentation originates in Fregean philosophy, where it is used to account for the sense, as opposed to the reference, of linguistic expressions. Several philosophers of mind make free and easy appeal to the idea that mental representations, and not just linguistically expressible thoughts, have modes of presentation. Even so, the distinction between the "sense" and "reference" of neural representations is an ad hoc construction without any independent justification. C&B try to motivate the use of modes of presentation by speaking of how the gustatory system might be thought to represent levels of sodium chloride (referent) via a "salty" mode of presentation (sense). However, this comparison is confusing, because the saltiness of sodium chloride is something experienced by an organism. There seems to be no obvious reason to suppose that there is a specific way that sodium chloride is presented to our sub-personal gustatory systems. By the same token, it is unclear what warrants assuming that a sub-personal neural system, such as the ANS, operates with a "mode of presentation," or how we would be in a position to know which particular "mode of presentation" such a system would employ if it did. Positing "modes of presentation" does a lot of heavy lifting for C&B, but their appeal to that technical notion seems to be a "just so" solution, motivated by philosophical need rather than justified by independent empirical considerations.

There may be reasonable grounds for distinguishing the different ways organisms experience worldly targets or the ways people variously represent the same extension. What is not clear is that C&B can innocently assume that modes of presentation operate at the neural level. Nor is it clear how they justify attributing the particular modes of presentation to the ANS that they do. After all, when presented with supposedly imprecisely represented collections, we do not experience them as imprecisely presented to us. Instead, it is simpler to assume that we are sensitive to numerical properties, just not optimally so (as one would expect given physiological constraints).

The problems with C&B's representationalism stem from their assumption that the ANS's *sole* function is numerical perception. In essence, they assume that the ANS is some form of number module. However, the evidence suggests that the neural system that houses the ANS is involved in a whole host of other capacities, including motion processing, mental imagery, working memory, and the control of visuo-spatial attention and pointing and grasping motions (Culham & Kanwisher, 2001; Gillebert *et al.*, 2011; Grefkes & Fink, 2005; Simon, Mangin, Cohen, Le Bihan, & Dehaene, 2002), in line with the predictions of Anderson's neural reuse theory (Anderson, 2014; Hutto, 2019; Jones, 2020; Penner-Wilger & Anderson, 2013). Once one gives up on the idea that the ANS is a system solely *for* dealing with number, the idea that its job is to *represent* number is far less tempting.

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A rational explanation for links between the ANS and math

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Abstract

The proposal by Clarke and Beck offers a new explanation for the association between the approximate number system (ANS) and math. Previous explanations have largely relied on developmental arguments, an underspecified notion of the ANS as an "error detection mechanism," or affective factors. The proposal that the ANS represents rational numbers suggests that it may directly support a broader range of math skills.

We applaud Clarke and Beck (C&B) for their convincing arguments supporting the presence of an approximate number system (ANS). Most importantly, we agree with their notion that the ANS represents numbers, not numerosities or non-numerical confounds, even if its representations can be derived from computations involving perceptual cues. The ANS has attracted increasingly more attention over the last decade as correlational and training studies suggest a link between the ANS and children's and adults' math abilities. Many (but not all) studies report that children and adults with greater ANS acuity tend to perform better on math assessments both concurrently and longitudinally (Chen & Li, 2014; Fazio, Bailey, Thompson, & Siegler, 2014; Schneider et al., 2017) and that training the ANS leads to improvements in children's and adults' math abilities (Bugden, DeWind, & Brannon, 2016). However, none of these studies have been able to provide a definitive mechanistic explanation for the association between the ANS and math abilities. Previous explanations have (1) largely relied on developmental arguments, (2) invoked the function of the ANS as an error-detection mechanism, or (3) cited possible motivational or affective factors.

Several different possibilities may explain the link between the ANS and math throughout development. On the one hand, it is possible that a more precise ANS may better support children's acquisition of exact number representations (Pinheiro-Chagas et al., 2014; Wagner & Johnson, 2011). For example, children's ability to map between symbolic and non-symbolic quantities is associated with their math achievement, suggesting that ANS representations are involved in the development of children's math skills via their associations with number symbols (Mundy & Gilmore, 2009). On the other hand, it is possible that a more precise ANS may serve as a foundation to understand ordinal relations between quantities and their relation to arithmetic operations, especially as children acquire these math skills (Libertus, Odic, Feigenson, & Halberda, 2016; Mussolin, Nys, Leybaert, & Content, 2016; Park, Bermudez, Roberts, & Brannon, 2016). For example, the ability to identify ordered sequences of Arabic numerals mediates the relation between the ANS and adults' mental arithmetic (Lyons & Beilock, 2011) and steadily increases in its role between first and sixth grades (Lyons, Price, Vaessen, Blomert, & Ansari, 2014).

Another explanation is that the ANS may provide a sense of certainty about number-related judgments or serve as an "error detection mechanism" providing rough estimates of arithmetic computations and aiding in the detection of gross miscalculations

(Baer & Odic, 2019; Vo, Li, Kornell, Pouget, & Cantlon, 2014). For example, individuals' ability to detect errors in symbolic arithmetic problems is related to their ANS acuity (Wong & Odic, 2021).

Finally, the ANS and math may be linked via motivational or affective factors. For example, greater ANS acuity in childhood may increase children's attention to number or engagement with math-related information (Libertus, 2019). Alternatively, greater ANS acuity may lead to greater confidence in mathematical reasoning (Wang, Odic, Halberda, & Feigenson, 2016) or poorer ANS acuity may lead to increased math anxiety (Lindskog, Winman, & Poom, 2017; Maldonado Moscoso, Anobile, Primi, & Arrighi, 2020; Maloney, Ansari, & Fugelsang, 2011).

Many of these explanations rest on the (albeit, implicit) assumption that the ANS represents natural numbers. As such, extant hypotheses cannot fully explain why the ANS may be correlated, for instance, with adults' performance on college entrance exams in math that require far more than whole number arithmetic (Libertus, Odic, & Halberda, 2012). Even if, for example, the ANS is involved in error monitoring during calculations, how could this system operate to detect errors in calculations that do not depend solely on positive integers? Clarke's and Beck's proposal that the ANS represents rational numbers opens up an exciting additional explanation which may provide a missing link in the theoretical pathway from non-symbolic number representations to math abilities. Specifically, their proposal that the ANS represents rational numbers would provide a compelling explanation of how the ANS may directly support a broader range of math skills that transcend the natural numbers and operations thereon, including fraction understanding and proportional reasoning.

However, as C&B mentioned, there is a dearth of research on non-symbolic ratio processing. Future research should test the sensitivity of the ANS to rational numbers and probe the relation between the ANS and the ratio processing system (RPS), which the authors argue is a component of the ANS. An initial step is to examine the associations between individuals' performance on a wide range of tasks tapping into the ANS and the RPS that have previously only been used in separate studies. Although some research has suggested that the ANS is recruited during tasks that require the RPS or proportional reasoning (Matthews & Chesney, 2015), no studies have explicitly established a correlation between the precision of these systems. O'Grady and Xu (2020) posit that children's proportional judgments of non-symbolic dot arrays are reliant on the ANS to represent discrete numbers, which are used to calculate probabilities. However, it is unclear whether the relational processing of whole numbers is supported by the ANS and/or facilitated by the RPS.

Extending beyond ratio processing, recent research on risky decision-making involving non-symbolic quantities demonstrates an association between adults' performance on tasks tapping the ANS and probability understanding (Mueller & Brand, 2018). For instance, individuals' non-symbolic quantity estimation relates to their abilities to estimate risks presented non-symbolically, and both of these abilities relate to adults' ability to transform and compare symbolic probabilities, an important aspect of math abilities beyond whole number operations (Mueller, Schiebener, Delazer, & Brand, 2018). Thus, Clarke's and Beck's view of the ANS may also provide an explanation for these findings and suggest further interesting research directions, including the development of probability understanding and its link to the ANS.

In sum, the proposal that the ANS represents rational numbers helps in further elucidating the link between the ANS and math abilities. This perspective opens up interesting new directions for future research, including probing the relations between the ANS and RPS as well as understanding the relations between the ANS and decision-making processes.

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Contents of the approximate number system

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Abstract

Clarke and Beck argue that the approximate number system (ANS) represents rational numbers, like 1/3 or 3.5. I think this claim is not supported by the evidence. Rather, I argue, ANS should be interpreted as representing natural numbers and ratios among them; and we should view the contents of these representations are genuinely approximate.

A natural view is that the approximate number system (ANS) represents… well, approximate number. That is, (i) its outputs are abstract, sortal-dependent, second-order representations, serving as answers to the question "how many?" (rather than, say, "how much?"); but (ii) these representations are semantically approximate: their contents are something like 13ish, and not so precise as 13, certainly not 13.7. This would be an especially good view for Clarke and Beck (C&B) to embrace. If they're right to reject the sensitivity principle then (i) and (ii) are consistent; there need be nothing oxymoronic about "approximate number."

Instead, however, they claim that ANS represents rational numbers, like 1/3, 3.5, and 2.75, a claim not easily squared with this natural view.

C&B's argument for thinking that ANS represents rational numbers is based on studies indicating a sensitivity not just to numerosities but also to ratios among these numerosities (e.g., Denison & Xu, 2014; McCrink & Wynn, 2007). These studies, however, don't support C&B's claim. It is one thing to say that ANS represents (approximate and natural) numbers *and ratios among them*, and quite another to say that it represents the rational numbers. The former is much more plausible than the latter. Among the few examples C&B give of non-natural rational numbers are 3.5 (sect. 7.3, para. 1) and 2.75 (sect. 7.3, para. 10), but they present no evidence that ANS represents such numbers, indeed no evidence that ANS represents *any* non-natural numbers greater than 1. No evidence suggests that ANS represents some aggregate as containing 3.5 items, or as containing 2.75 times as many yellow items as red ones.

C&B should have said that ANS employs contents like 1/3ish in addition to 13ish. By claiming that ANS represents rational numbers, like 2.75, rather than approximate ratios, C&B seem to attribute to ANS *greater* precision than had it merely represented natural numbers, when what it needed was *less*. They address this tension by insisting that, although ANS represents precise quantities, it represents them imprecisely; it's the representation that's imprecise, not what it represents.

There are two ways to read this claim.

On the first interpretation, C&B are saying that although an ANS representation has precise truth conditions, it is, perhaps because of the physical or syntactic properties of the representational vehicle, easily confused for some other representation, with similar but distinct truth conditions. This is a kind of "vehicular imprecision" rather than "semantic imprecision." A vehicular-imprecision-with-semantic-precision view makes the same behavioral predictions as a semantic imprecision view, but the former implies that, because the ANS truth-conditions are precise (e.g., 13, rather than 13ish), ANS representations will be in error very often, perhaps much more often than not. (Even if we discriminate 13 from 14 at levels somewhat better than chance, and even if that shows that we're applying 13 contents more than half of *those* times, we would need to discriminate 13 from the disjunction 11-or-12-or-14-or-15 [etc.] at better than chance in order to be applying 13 correctly more often than not.) A semantic imprecision view will ascribe much less error, as a 13ish verdict is presumably true of a 14-item array. Everything else equal, a theory that ascribes less representational error is to be preferred over one that posits more, and C&B, on this interpretation, seem to be ascribing error quite gratuitously by insisting on precise contents.

A second interpretation sees C&B espousing semantic imprecision after all, allowing them to embrace the natural view I started with. Here, the precision lies not in the representation, but in the representational "target" (Cummins, 1996; C&B call it "referent," [sect. 2.2; note 1; sect. 6, para. 10]), that is, the thing to which the representation is applied, and of which that content is predicated. Although Jones has a precise weight, we might represent that weight imprecisely (sect. 5.1, para. 2). This avoids the problems for the first interpretation, but it's no longer the claim that ANS represents rational numbers, in the only sense that could be relevant. Suppose I misrepresent a dog as a cat. I thereby apply a cat representation to what is, in fact, a dog; that dog is the target of this cat. If we were to say – misleadingly, with C&B – that the dog was the "referent" of cat, then we would be tempted to claim that cat represents (/means, /refers to) dogs, but this is clearly false, at least on any standard construal. Yet it's exactly this reasoning that C&B use to argue that ANS represents rational numbers.

If C&B hold merely that rational numbers are *targets* of ANS representations, then it's unclear where they disagree with Carey (2009) and Núñez (2017), both of whom are surely aware that aggregates typically contain precise numbers of items and thus agree that ANS represents a precise thing imprecisely in this sense.

But anyway, C&B can't – or can't *only* – be saying that we apply a 16ish to instances of 16.29 in the world. They're saying we apply 16.29s, if they're saying anything. The claim that ANS represents rational numbers is supposed to be explanatory, but it can't be explanatory if it's only a claim about the targets of ANS representations and not the contents. It is completely unexplanatory to claim that ANS represents a 2:1 ratio of yellow to red items, if that claim is only a statement about the stimulus and leaves open all possibilities about how ANS represents that stimulus.

Because they reject the sensitivity principle, however, C&B didn't need any of this trouble. If sensitivity is false, then precision never needed to figure into C&B's account. They could have simply claimed that ANS represents approximate natural numbers and ratios among these.

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Weighted numbers

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Abstract

Clarke and Beck (C&B) discuss in their sections on congruency and confounds (sects. 3 and 4) literature that has challenged the claim that the approximate number system (ANS) represents numerical content. We argue that the propositions put forward by these studies aren't that far from the indirect model of number perception suggested by C&B.

In sections 3 and 4 of their proposal, Clarke and Beck (C&B) discuss a body of literature challenging the claim that the approximate number system (ANS) represents genuine numerical content coming from studies demonstrating the presence of congruency effects and interference from non-numerical confounds in non-symbolic number comparison (e.g., Gebuis, Cohen Kadosh, & Gevers, 2016; Gebuis & Reynvoet, 2012; Smets, Sasanguie, Szucs, & Reynvoet, 2015). We argue that in their current paper, C&B somewhat misinterpreted these articles' key message. In fact, the propositions put forward by these studies are very similar to the indirect model of number perception C&B suggest.

To begin with, the abovementioned studies do not claim that the ANS "[…] merely represents a *mishmash* of non-numerical magnitudes" (C&B, sect. 4, para. 11; emphasis added) – at least not in the way C&B conceive the ANS. These studies contested the what is called here, direct model of number perception – the idea that "number" is a primary feature of a set, which can be directly perceived from the environment. Instead, these studies argued that "[…] number judgements are based on the *integration of information* […]" (Gebuis & Reynvoet, 2012, abstract; italic added). In our view, this idea is very similar to the indirect model of the ANS C&B appeal to themselves.

Researchers in numerical cognition are well aware that the number of a set is confounded with non-numerical magnitudes such as size in the visual modality and pitch length in the auditory modality. Consequently, various algorithms have been developed to control these non-numerical magnitudes by making them uninformative for the decision (e.g., Gebuis & Reynvoet, 2011; Halberda, Mazzocco, & Feigenson, 2008; Marinova, Sasanguie, & Reynvoet, 2021). Nevertheless, even when these confounds are accounted for, they still affect participants' performance in non-symbolic number comparison and lead to congruency effects (i.e., lower accuracies when numerical and non-numerical magnitudes conflict; see Reynvoet et al., in press; Smets et al., 2015). Some studies even observed that the size of the congruency effects depends on the interrelation of different non-numerical magnitudes. For instance, the congruency between one non-numerical magnitude (e.g., the convex hull) and number and another non-numerical magnitude (e.g., size of the individual dots) and number can result in an additive effect. Alternatively, they can also cancel out each other (Gebuis & Reynvoet, 2012). Based on these findings, researchers proposed that participants integrate the information from multiple visual cues into one weighted sum (Gebuis et al., 2016; see also Picon, Dramkin, & Odic, 2019).

The idea of integrated information is also very similar to the excellent example of the representation of depth C&B describe in their article. Here, the authors argue that the representation of depth is constructed based on various visual inputs, which can be weighted differently depending on the context (e.g., some inputs may be less informative in particular situations and given less weight). Therefore, the main difference between our previous study and the indirect model proposed by C&B does not lie in the pre-assumed underlying perceptual and cognitive processes. Rather, it lies in what one considers a "number." In our view, the weighted sum of the visual inputs can be regarded as a representation of number, similar to the representation of depth (see also Halberda, 2019 who describes a similar position from a reductionist/empiricist and rationalist point of view). Our previous study shows that in many situations, the representation of number on which numerical decisions are based is influenced by the (incongruent) non-numerical magnitudes. As C&B and some proponents of the direct model of number perception rightfully argue, this interference may arise at a response stage (similar to a classic Stroop effect). In contrast to this claim, in their study, Picon et al. (2019) demonstrated that interference between non-numerical magnitudes and number occurs early in the processing stream – an observation rather in line with an indirect account of number perception. It is worth noting that the idea of "weighted" number representation does not entirely rule out the possibility of a direct model of number perception. Concretely, a substantial body of literature suggests that automatic and direct extraction of number is possible (e.g., Burr & Ross, 2008; Van Rinsveld et al., 2021). The latter case is especially plausible in adult participants because of the extensive focus on "number" in the educational curricula.

Finally, instead of remaining agnostic to one of the most crucial questions in numerical cognition about whether the number is extracted directly or indirectly, we believe it is possible to reconcile the direct and the indirect models of number perception. Concretely, whether the numerical decision would be based on direct and automatic extraction of number or indirect weighted number representation possibly depends on various factors such as developmental differences (e.g., Piazza, De Feo, Panzeri, & Dehaene, 2018), stimulus set (e.g., Reynvoet et al., in press), and so on.

In sum, in this commentary, we argued that previous studies describing congruency effects resulting from contrasting numerical and non-numerical cues could be easily reconciled with the indirect model of the ANS as proposed by C&B. That is, multiple sources of information may be integrated into a higher-order representation of number. We also acknowledged that in some circumstances, the number might be extracted directly. Whether the number will be processed directly or indirectly, as a weighted sum of visual inputs, depends mainly on the individual and the context in which the numerical decision arises. Finally, it is also worth noting that, thanks to its exceptionally well-presented framework, the article by C&B provides an excellent starting point to disentangle further the conditions under which direct and weighted number representations occur.

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The number sense represents multitudes and magnitudes

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Abstract

Clarke and Beck's view that numbers are *both* second-order and sensible is based on an empirically dubious claim, which is required to show that what they call the "weak sensitivity principle" is satisfied. The explanatory benefits that they say are gained by positing a sensory relation to numbers are also gained by positing such a relation to *multitudes* of objects.

Clarke, Beck, and I all agree that finite cardinal numbers "have a 'second-order character' that non-numerical quantities lack" (sect. 5.3, para. 4). This means that numbers are properties that apply to objects *relative to a given sortal-kind* whereas first-order properties apply to objects tout court. We disagree on the question of whether approximate number system (ANS) explanations of our quantitative judgments require positing a sensory relation to numbers (cf. Azzouni, 2010). To me, this idea is very odd because second-order entities are no part of the sensible realm, and so have to be grasped by thinking rather than sensing (Marshall, 2017). Why do Clarke and Beck (C&B) find it necessary, or even remotely plausible, to say that numbers are *both* second-order and sensible?

They seem to think that plausibility is inherited from Tyler Burge's theory that the ANS operates over objects that have already been individuated by perceptual representations of kinds that Burge calls "perceptual attributives" (2010). These attributives are supposed to be the perceptual correspondents of concepts, elements of perceptual states that represent properties and kinds. The problem is that an examination of Burge's work does not reveal any *evidence* that the ANS operates in this way. Rather, Burge simply draws on his theory of perceptual attribution to assert *without empirical argument* that the ANS can carry out processes of individuation and enumeration that are analogous to transitive counting. Although not as circular as positing a homunculus who can count transitively, this does smuggle conditions required for counting into the description of the ANS. Such speculation may show that perception of numbers is *possible*; but it does not constitute evidence that the ANS *actually* operates in this way. (By the way, if there is no such evidence, this would weaken C&B's criticism of Burge's proposal that the ANS represents the magnitudes and ratios described by Eudoxus, a criticism that appeals to the fact that the ANS represents second-order entities.)

C&B offer *some* evidence that the ANS draws on perceptual attributives – namely, the dumbbell studies, on which they have to hang *an awful lot*. This is because as I show in my (Marshall, 2018), there are various essential properties of numbers that are not represented by the ANS. Furthermore, what C&B call the "weak sensitivity principle" is satisfied only if the ANS can represent some essential property of numbers; further, they argue that it does so in virtue of representing their essentially second-order nature. They also admit that if the principle is *not* satisfied, then it is unclear "what else could make it the case that [numbers are] being represented rather than some other entity." Hence, they have to hang so much on the dumbbell studies.

C&B avoid committing themselves to Burge's model by distinguishing Marr's computational and algorithmic levels of explanation, allowing that the ANS could function at the former level to represent numbers *qua* second-order entities, while leaving open exactly *how* this is done at the algorithmic level. The problem is that *what* is represented places constraints on *how* it is represented and so one cannot offer a theory at the computational level in complete abstraction from the algorithmic level (cf. Burge, ibid, p. 93, fn. 43). As things stand, the situation seems to be that if we could sense numbers, then the ANS would have to operate as Burge says; but there is little evidence that it does. To defeat this argument by showing how else the ANS could operate, more needs to be said about how their alternative "indirect model" of the ANS could generate representations of second-order numbers from representations of "a mishmash of non-numerical magnitudes" (sect. 4, para. 11).

Developing this line of thought, C&B argue that positing a sensory relation to numbers would be needed to unify ANS explanations of behavior and to explain the common function that our sensitivity to the aforementioned mishmash would be serving. However, the same explanatory benefits are gained by positing a sensory relation to *multitudes* of sensible objects, where these objects *taken collectively* possess non-numerical but quantitative properties from which an approximate representation of their multitude might be extracted by the ANS.

They might object that I cannot substitute talk of multitudes for talk of numerosities, because only the latter are, like numbers, second-order: just as we speak of the number of *F*'s, so we speak of the numerosity of *F*'s. I respond that the relevant sortal-kind of the objects making up the multitude can be discerned from the context if necessary. When I sense that there are two oranges, I sense an orange, another, and no more. When I sense that there are just as many oranges as there are apples, I sense an orange, another, and no more, an apple and another and no more, and match them up so that none are left over. If I cut off half an apple, leaving the remainder, I sense that there is an apple and a half, which is to say one and a half apples worth of apple, as well as three and half fruits worth of fruit. Therefore, this proposal also encompasses what C&B (mistakenly) call our sense of rational numbers. In any case, the sortal-kind is the natural analog of a unit of measurement that is provided by and can be discerned from the context. The connection to numbers is as follows:

The sensible world contains multitudes from which cardinal numbers are abstracted. The latter are then applied to the world *as it is organized by sortals*. Numbers are not simply properties of the sensible world but properties of the world *as it is organized* (cf. Gaifman, 2005). The sensible world also contains magnitudes (as well as natural analogs of units of measurement), which fall into ratios from which rational numbers are abstracted. The ANS does not represent numbers; but, in virtue of approximately representing multitudes and ratios of magnitudes it is correlated approximately with numbers, because numbers by their nature apply to multitudes relative to sortals as well as to magnitudes and their ratios.

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Numerosities are not ersatz numbers

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Abstract

In describing numerosity as "a kind of ersatz number," Clarke and Beck fail to consider a familiar and compelling definition of numerosity, which conceptualizes numerosity as the cognitive counterpart of the mathematical concept of cardinality; numerosity is the magnitude, whereas number is a scale through which numerosity/cardinality is measured. We argue that these distinctions should be considered.

The difference between cardinality and number is often overlooked by those who defend the numerical character of the so-called number sense. Clarke and Beck are no exception. This can be gathered from their description of numerosity as "a kind of ersatz number" (sect. 6, para. 3). Although they correctly point out that approximate number system (ANS) researchers rarely state explicitly what they mean by "numerosity," they miss those who do. This term was introduced by the psychophysicist Stevens (1939/2006) to disambiguate the term "number," ambiguously used to designate a scale for the measurement of cardinality and the very magnitude this scale measures. Stevens reserved the word "number" for the first use, and introduced "numerosity" for the second. Consistent with Stevens's definition, in the contemporary literature one finds "numerosity" defined as a synonym for cardinality (e.g., Butterworth, 2005, p. 3; Nieder, 2016, p. 366; Piazza & Izard, 2009, p. 261).

Stevens's distinction between number and numerosity is not arbitrary. In mathematics, "cardinality" and "number" refer to different concepts. Cardinality refers to set size, whereas numbers provide a way of determining and expressing set size; it is a trivial observation that the cardinality of a set can be determined and expressed without even mentioning numbers. Indeed, suppose that we want to know whether the cardinality of the set of people in a room is equal to the cardinality of the set of chairs in that room. We need not count people and chairs; we can just ask people to sit down, one person per chair. If no person remains standing and no chair remains empty, we conclude immediately that both sets have the same size. If someone remains standing, then the set of people is larger than the set of chairs; if chairs remain empty, then the opposite is the case. No number is involved in this procedure; it can be carried out recruiting only the notion of one-to-one correspondence or equinumerosity, which, despite its name, is defined without invoking numbers (Enderton, 1977, p. 129). In principle, we can use any set as a "yardstick" to evaluate and express cardinalities. For example, let *P* be the set {a, b}; we can say that the cardinality of the set of authors of this commentary is equal to *P*'s cardinality; that the cardinality of the set of planets in our solar system is larger than *P*'s cardinality; and so on. In sum, cardinality is the magnitude, whereas number is a scale for the measurement of this magnitude. In principle, there can be scales other than numbers, such as the one based on *P*.

C&B compare perception of "number" with perception of other magnitudes, such as distance. They claim that, if we acknowledge that the visual system represents distance, we should also acknowledge that the ANS represents number. But this is where the confusion between cardinality and number misleads them. Although the visual system represents distance, it does not represent meters or feet (scales for the measurement of distance); by the same token, we may acknowledge that the ANS represents *numerosities*, but this does not mean that it represents number. Scales are not the sorts of things that can be perceived. At most, it may be that subjects use numbers as a mental scale to evaluate numerosity, but they certainly do not *perceive* numbers as such (which arise from applying a specific scale to magnitudes).

We, thus, have one way of resisting C&B's attack on the argument from imprecision. Whatever it is that the ANS uses to measure numerosity, it is imprecise. Therefore, it is unlikely that the ANS uses numbers, because by definition numbers constitute an exact scale for the measurement of cardinality (notice that there may be inexact scales; the one based on *P* above is one example; an approximate accumulator is another).

But is it correct to say that the ANS *represents* numerosities imprecisely? As C&B notice, there is a crucial difference between (what they call) number (and we call numerosity) and other magnitudes such as distance. Although the latter are "first-order," or intrinsic, properties of the environment, "numerical quantities are higher order in that they can only be assigned relative to a sortal – a criterion for individuating the entities being counted" (sect. 5.3, para. 4). C&B are to be praised for highlighting this distinction, which is often overlooked in the literature. But it has a consequence they do not seem to consider: In contrast to distances, there are no numerosities "out there" to be perceived, because numerosities emerge only after an agent has adopted a given sortal. We can say that distances are imprecisely represented by the visual system because there seem to be exact distances in the external environment, regardless of some agent noticing them. But we cannot say the same about numerosities, because there are no numerosities out there, inexact or otherwise, to be represented: they emerge relative to a given sortal. As C&B acknowledge, "numbers [numerosities] … enter into contents via property attribution, not as objects of perception" (sect. 2.2.3, para. 1). But attribution and representation seem to have different directions of fit: Attribution goes from subject to stimulus, whereas representation goes from stimulus to subject. Ultimately, the notion of representation that they adopt seems in tension with their own higher-order understanding of numerical quantities.

Finally, if numerosity is understood as pertaining to cardinality, it is also incorrect to say that subjects *attribute* numerosities to stimuli, because cardinality is an exact property by definition. The property the ANS assigns to stimuli, however, is vague. Stevens introduced the term "numerousness" to designate this property: "a property or attribute which we are able to discriminate when we regard a collection of objects" without counting (Stevens, 1939/2006, p. 1). This term has (regrettably) fallen into disuse, but we think that accounts of the so-called number sense would still benefit from a clear distinction between number (a scale), numerosity (a precise property), and numerousness (a vague property attributed in perception) (dos Santos, 2021).

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The perception of quantity ain't number: Missing the primacy of symbolic reference

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Abstract

Clarke and Beck's defense of the theoretical construct "approximate number system" (ANS) is flawed in serious ways – from biological misconceptions to mathematical naïveté. The authors misunderstand behavioral/psychological technical concepts, such as numerosity and quantical cognition, which they disdain as "exotic." Additionally, their characterization of rational numbers is blind to the essential role of symbolic reference in the emergence of number.

The target article by Clarke and Beck – written with an unnecessary condescending tone – is flawed at many levels, from biological misconceptions to mathematical naïveté, and exhibits serious inconsistencies. Here, we only address those most crucial.

First, the article lacks clarity regarding the central concept of "number." The authors open by assuming that humans and other animals possess an "*approximate number system* (ANS) that represents number," but never provide a working definition of *number*. They simply take number as pre-given and unproblematic. For a highly polysemous term like "number," this presents major problems, especially when the goal is to defend the ANS hypothesis and claim that it (also) represents rational numbers. Little clarity can emerge from arguments that blur perceptual, linguistic, and conceptual dimensions of quantity treatment, all falling under the umbrella of "number." The authors' use of mathematical concepts such as "natural numbers" and "rational numbers" doesn't help either. These are technical concepts that refer to infinite sets governed by specific axiomatic systems which, among others, determine their elements via a categorical membership relation. One entailment is that, for a given set, no element is more familiar or typical than another one – mathematically, 38,980,254,332,198 is "as natural" as 2, and 1/2 "as rational" as 577843/97816. Although the authors mention that the ANS does not represent *every* natural (or rational) number "or even *most*" rational numbers (whatever "most" may mean in the case of this infinite set, dense in the real numbers), they provide no theory of *which* rational numbers – and by means of what criteria – are supposed to be represented by the ANS (other than saying that they are "of a familiar sort"). A more appropriate title for their article would, thus, be "The number sense represents *some* rational numbers (but it is unclear which)." The authors' confusing use of mathematical concepts and terminology (e.g., stating that "real numbers are continuous") just makes things worse.

Second, the authors erroneously criticize behavioral/psychological technical concepts, such as "numerosity," which they disdain as "exotic." Although they are right in that the term "numerosity" has been misused in the numerical cognition literature (Davis & Pérusse, 1988; Núñez, 2017a), they ignore that this term was coined by the psychophysicists of the 1940s who were seeking for conceptual clarity when investigating the problem of scales of measurement of psychological magnitudes (Stevens, 1939/2006, 1951). Renowned experimentalist S.S. Stevens referred to numerosity as "a property defined by certain operations performed upon groups of objects" (1939/2006, p. 23), with the goal of evaluating their numerousness by means of which an experimenter ultimately establishes the cardinal attribute of physical collections of objects. Contrary to the authors' claim, numerosity was not coined as an "exotic substitute for number," but as a careful attempt to disentangle the abstract conceptual content of "number" from the degree to which an experimenter could reliably evaluate the attribute of numerousness of stimuli. Thus, the sound and well-defined statement "five is a prime *number*" was never meant to (and cannot) be substituted by "five is a prime *numerosity*."

The authors also brush off the term *quantical* (Núñez, 2017a) as "exotic," misconstruing its meaning and its theoretical entailments. They erroneously characterize it as a *noun* ("quanticals") serving "as a substitute for number" (with "mysterious properties") whereas, in fact, "quantical" was proposed as an *adjective* – in contrast to "numerical" – meant to characterize some biologically endowed forms of non-symbolic quantity-related cognition and capacities. The authors also misrepresent the quantical–numerical distinction as about "imprecision," conceived to critique the ANS hypothesis on this ground. But the essence of the distinction is about the capacity of *symbolic reference* (Deacon, 2011) – rich in humans and largely absent in nonhuman animals – which the authors fail to appreciate. Subitizing, for example, is a form of quantical (non-symbolic) cognition, yet still precise. The quantical–numerical distinction is not in the business of making claims about the ANS representing anything (let alone the authors' imagined "quanticals"). Rather, by pointing to the symbolic reference property inherent in number (but not in purely perceived quantities of items) it leads to the critique that the construct "ANS" teleologically puts number (hence, the "N") directly in the category of what is biologically endowed, without symbolic (and therefore cultural) mediation. Attacking the "quanticals" strawman to defend the ANS hypothesis is, therefore, fallacious.

Third, the authors' arguably only novel claim is that the ANS represents rational numbers because it "represents *ratios* among positive integers." Numerically, however, ratios presuppose a binary *arithmetic operation* (division) which, beyond numbers proper, would have to be biologically endowed and implemented *qua arithmetic operation*, a biological no-go. Moreover, statements such as "while the ANS probably represents 2.5 and 2.75, there is no evidence that the ANS can represent 2.7452294861" are theoretically untenable. There is no evidence, or reason to believe, that the hypothesized ANS (or any biological system) "represents" numbers in base 10, which would render 2.75 "more representable" and familiar than 2.7452294861 (presumably because of its shorter decimal expansion). Indeed, 2.75 expressed in, say, base 7 yields $2.51515151\ldots_{(7)}$, with an infinite decimal expansion. The taken-for-granted *expression* of rational (or any) numbers reveals the crucial miss of symbolic reference in the argument. It prevents the authors from seeing that (1) psychophysical perception of quantities of items and (2) the numbers obtained by the measurement of stimuli's attributes (loudness in decibels, relative quantity in numerical ratios, etc.) are fundamentally different phenomena. The former – shared by many animal species – evolved largely via natural selection, the latter requires symbolic reference implicated in language and specific cultural practices on the part of the schooled experimenter or philosopher, and has evolved via cultural evolution (Beller & Bender, 2008; d'Errico *et al*., 2018; Gray & Watts, 2017; Núñez, 2017a, 2017b). The evolution of such bio-cultural underpinnings of quantification and number is the subject matter of exciting new areas of multidisciplinary research such as those implemented in QUANTA, an endeavor supported by the European Research Council (Barras, 2021). Essential in this enterprise is the recognition of the primacy of symbolic reference in the evolution of cognitive tools for quantification.

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Unwarranted philosophical assumptions in research on ANS

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Abstract

Clarke and Beck import certain assumptions about the nature of numbers. Although these are widespread within research on number cognition, they are highly contentious among philosophers of mathematics. In this commentary, we isolate and critically evaluate one core assumption: the identity thesis.

Clarke and Beck (C&B) seek to defend the thesis that states of the approximate number system (ANS) represent, or refer to, "not only natural numbers (e.g., 7), but also non-natural rational numbers (e.g., 3.5)." In doing so, however, they rely on some highly controversial, and largely undefended philosophical assumptions.

The assumptions we have in mind arise most clearly in section 2.2, where C&B mention an apparent puzzle for their view: If the ANS often operates perceptually in order to represent numbers, and if numbers are abstract objects, then how could numbers feature in the contents of such perceptions, if we cannot generally perceive abstract objects? In response, they adopt a "standard picture" of perception on which perceptual states both refer to concrete objects and attribute (abstract) properties to them. On this view, "abstract objects enter perceptual content through the attribution of properties, not through reference to objects." For example, perceiving a red apple involves referring to a concrete apple and ascribing the abstract property of redness to that apple. Similarly, perceiving that there are four apples involves referring to a set of apples and attributing an abstract cardinality property to it. In this way, C&B suggest that states of the ANS refer to numbers "by enabling numbers to enter into contents via [cardinality] property attribution."

The present view carries two significant, although controversial, implications familiar from a minority view within the philosophy of mathematics known as *empiricism* (Kim, 1981, Maddy, 1990). First, because sets are the objects referenced by certain perceptual states, those sets must be concrete. The likely view, following Kim and Maddy, would be that *impure sets* – sets whose members are concrete urelements – are themselves concrete. This is controversial, of course, as the predominant view is that sets are uniformly abstract. But perhaps C&B are talking loosely. By "set," perhaps, they intend some more intuitive notion of collection, which might be elucidated in different ways, for example, mereologically.

The second, more pressing implication is what we call *the Identity Thesis*:

(IT) Natural numbers are identical to cardinality properties.

This view is philosophically controversial, linguistically problematic, and threatens an explanatory regress.

First, IT is philosophically controversial, in part, because it presupposes a *cardinal conception* of the naturals, whereby natural numbers are essentially the sorts of things we can use when counting. However, this is not the only – or even dominant – characterization available within the philosophy of mathematics. There are also *structuralist* characterizations (Shapiro, 1997), *ordinal* characterizations (Linnebo, 2018), and *geometric* characterizations (Tennant, 2009), for instance. Furthermore, even among advocates of the cardinal conception IT is controversial because properties are *intensional* entities, whereas the naturals on all prominent cardinal characterizations are *extensional*, identified with (finite) sets or classes (Frege, 1950; Hale and Wright, 2001; Maddy, 1990; Russell, 1903).

Second, IT is linguistically problematic because the presumption of its truth appears to yield numerous false semantic predictions. Consider the following semantic contrasts inspired by Moltmann (2013) and Snyder (2017):

(1) (a) The (??rational) number of women at the party is {four/??the number four}.
(b) The (rational) number Mary is writing about is {four/the number four}.
(2) (a) How many women are at the party? {Four/??The number four}.
(b) Which (rational) number is Mary writing about? {Four/The number four/??The number of women}.

(3) (a) The number {of women/??four} is expanding rapidly.
(b) The number {of women/??four} exceeds that of the men.

Plausibly, "the number of women" refers to a cardinality property, whereas "the number four" refers to a number (Snyder, 2017). Thus, *pace* IT, (1)–(3) strongly suggests that cardinality properties are not numbers.

Furthermore, if this is so, then contra C&B, "numerosities" are plausibly neither "exotic" nor "recherché." Rather they are, as Butterworth (2005) and others appear to assume, cardinality properties – entities for which we already possess mathematically and linguistically well-understood theories (Scontras, 2014; Snyder, 2017).

Finally, even assuming IT, we doubt that, when combined with C&B's presumed account of perception, it resolves the original puzzle of how numbers, qua abstracta, could be referenced by perceptual states. In order to see why, consider what we take to be the two main ways of elaborating the proposal.

The first maintains that to attribute a property *just is* to reference that property. Thus, numbers enter into perceptual contents in virtue of consisting of two components, a set and a number, both referenced in perception. However, as should be clear from the following examples, this conflates predication and reference:

(4) (a) The apples are <u>four</u> (in number).
(b) <u>Four</u> is an even number.

In (4a), "four" functions as a predicate, whereas in (4b) it functions as a singular term, that is, a referential-type expression. Generally, predicates and singular terms have different semantic types, and thus semantic values. The present suggestion conflates those.

The second option characterizes predication as a relation between objects: "$F(a)$" is true just in case a instantiates F-ness, where "F-ness" names a property, viewed as an abstract object (Chierchia, 1985). Thus, numbers enter into perceptual contents in virtue of those contents consisting of three components: a set, the instantiation relation, and a number.

This proposal avoids conflating predication and reference, but only at the cost of an explanatory regress. On the present suggestion, the attribution of cardinality properties in perception requires reference to two objects, related by instantiation: a concrete set and an abstract number. But now the original problem recurs: Unless reference to numbers, qua abstracta, was possible, numbers couldn't feature in perceptual contents.

In the forgoing, we sought to cast doubt on the plausibility and explanatory utility of IT.

In doing so, however, we do not intend to suggest that C&B are idiosyncratic in their adoption of this assumption. On the contrary, we suspect that IT is implicit in much research on number cognition. If this suspicion is correct, then a significant upshot of our discussion is that this and related assumptions merit sustained critical scrutiny.

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Distinguishing the specific from the recognitional and the canonical, and the nature of ratios

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Abstract

There are three independent properties of a mode of presentation (m.p.) of a number: being specific; being recognitional; and being canonical. A perceptual m.p. of the form *that many Fs* is specific although it is neither recognitional nor canonical. The literature has not distinguished noncanonical from nonspecific m.p.s of numbers. Ratios are fundamentally ratios of magnitudes.

Clarke and Beck (C&B) contrast the representations of the approximate number system (ANS) with precise number concepts. Now, humans can employ a perceptually based way of thinking of a number *that many Fs*, as in *that many circles, that many toys*. *That many Fs* refers, in context, to a particular natural number. *That many Fs*, employed in a given context, does not itself supply information about which number it refers to as given in Arabic or any other canonical system for referring to natural numbers. *That many Fs* is, nevertheless, specific. It does not leave it open whether the reference is possibly a range of numbers, or a probability or confidence distribution over them. Similarly, *that length*, a mode of presentation ("m.p.") of a particular length as given in perception, does not give information about the length in any particular units. *That many Fs* and *that length* are specific, but not canonical.

A subject thinking of a number as *that many Fs* may or may not be able to recognize that number, so given, again. If the subject is a normal human, and the reference is to the number four, there will be a corresponding recognitional capacity. But there does not need to be for *that many Fs* to refer to in a given perceptual context.

This is an instance of an entirely general phenomenon in perception. A person without absolute pitch may hear a note, given under a perceptual m.p. *that note*. This is a specific m.p., but the subject may not be able to recognize the note again 10 min later. The distinction between discrimination and recognition applies to *that many Fs* just as it applies in elsewhere in perception. Therefore we must also distinguish the specific from the recognitional.

In the case of the perception of pitch, we can conceive of someone who can recognize the same pitch again over longer intervals but does not use any particular canonical system of representations (as middle C, D, E, F,…). Hence, we should further distinguish the recognitional from the canonical. Similarly for numbers: A subject may be able to recognize the number in question in uses of *that many Fs*, when the numbers are small, without employing a particular system such as the Arabic.

By contrast with C&B's claims about the m.p.s of an ANS, use of an m.p. *that many Fs* is sensitive, in matters of discriminability, to what individuates that number. Any natural number *n* is individuated by the condition for there to be *n* Fs, for arbitrary F. The successor relation in which a number stands is consequential upon this condition, rather than being fundamentally individuative of the number itself. To be sensitive to the condition for there to be *n* Fs does not at all require the subject to think of *n* as the successor of some other number.

Therefore, we need to distinguish an ANS from a noncanonical number system (NNS) which uses m.p.s of the form *that many Fs*. The modes of presentation employed in an ANS, as opposed to an NNS, are those expressed by *roughly that many Fs*. M.p.s of the form *roughly that many Fs* are genuinely unspecific, unlike those of the form *that many Fs*.

It is an empirical question whether a creature is using an ANS or an NNS. The two hypotheses are often not clearly distinguished. In both cases, the mental representations of number may involve analog elements. I hope for further empirical and conceptual work on the issues.

C&B also see subjects as "deriving ratios (hence rational numbers)" from representations of natural numbers in the ANS. A rival approach – originating in classical Greece – regards ratios as fundamentally ratios of magnitudes (such as distances, durations, and areas), of which ratios of natural numbers are one special case. Under this rival treatment, the "initial identification" (sect. 7.1, para. 1) of $\sqrt{2}$ is not quite what constituted a major discovery. There is clearly a ratio that is the ratio of the length of the hypotenuse of a unit right-angled triangle to the length of one of its other sides. That was not the major discovery. The discovery was rather that this ratio is not the ratio of any pair of natural numbers. It was not an option to say that there is no such ratio. The conception of ratios as ratios of magnitudes serves all comers. The discovery is that some such ratios are irrational.

Subjects can perceive the ratio of the lengths of one pair of objects as the same or as different from, the ratio of the length of a second pair of objects. The mental representation of ratios is important not only in the probabilistic examples the authors cite, but also in such decisions as foraging (ratio of food covering in a given area), and choice of a mate (ratio of size of body parts in a healthy conspecific). In such examples as these, the ratios may or may not be the ratios of natural numbers of entities of a given kind.

To think of something as a ratio, even a ratio of natural numbers, is one thing. It is another to represent this ratio mentally as a single rational number. The "converging evidence" cited by the authors (p. 42) does support the claim that organisms compare ratios of positive integers. It is a further claim, and would need additional evidence, to support the thesis that these ratios are coded as a single rational number, integrated into a single rational number line. Appreciation that 6:4, 12:8, and 3:2 are all the same ratio is not yet encoding that ratio as a single number 1½. *That ratio* is also a perceptually based way of thinking of a ratio, and, analogously to *that many Fs*, may not involve thinking of its referent as having a certain position in a rational number line.

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The approximate number system represents rational numbers: The special case of an empty set

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Abstract

We agree with Clarke and Beck that the approximate number system represents rational numbers, and we demonstrate our support by highlighting the case of the empty set – the non-symbolic manifestation of zero. It is particularly interesting because of its perceptual and semantic uniqueness, and its exploration reveals fundamental new insights about how numerical information is represented.

We agree with Clarke and Beck (C&B) that the approximate number system (ANS) genuinely represents numbers of a familiar sort, including natural and rational numbers. We systematically demonstrate our support based on the special case of an empty set – the non-symbolic manifestation of zero. An empty set is particularly interesting because of its perceptual and semantic uniqueness (e.g., Nieder, 2016; Zaks-Ohayon, Pinhas, & Tzelgov, 2021a, 2021b), which makes it a special case study in favor of the claim that the ANS represents rational numbers.

C&B reject arguments relating to congruency, confounds, and imprecision that claim the ANS fails to represent numbers. As expected, and in accord with the vast literature on the ANS, their position is supported by data and examples that are based on the use of non-null quantities that, regardless of their specific numerical values, are presented as "something" (i.e., at least one object). This is not the case for an empty set, however, which corresponds to null quantity or "nothing," and is presented as a form without content. In turn, when a frame containing an array of dots is compared with an empty frame, the perceptual uniqueness of the latter creates built-in, unavoidable *confounds* between the stimuli's discrete dimension – numerical magnitude – and its continuous non-numerical dimensions. Indeed, five continuous non-numerical stimuli dimensions were reported to interact with the numerical magnitude: the convex hull, the total surface area occupied by the dots, the density of the dots, the dot diameter, and the total dot circumference (for reviews, see Gebuis, Cohen Kadosh, & Gevers, 2016; Leibovich, Katzin, Harel, & Henik, 2017). In comparison with empty sets, each of these continuous non-numerical stimuli dimensions is confounded with the numerical magnitude in a *congruent* manner. For instance, the total surface area occupied by dots will always be larger in the non-empty set that contains one or more dots than in the empty set that does not, consistent with the non-empty set being numerically larger. Because these confounds are inherent to comparisons between empty and non-empty sets, they cannot be experimentally manipulated or controlled for. Moreover, contrary to the processing of non-empty sets, it would be hard to argue that the ANS represents an empty set *imprecisely* because if the frame is empty, it contains no objects and there is nothing to enumerate or estimate. In that sense, the perceptual prominence of empty sets presumably leads to a precise numerical evaluation that resembles subitizing (Kaufman, Lord, Reese, & Volkmann, 1949), the quick and accurate identification process for a small number (from 1 to 4) of objects.

Despite the inherent confounds in comparison with empty sets and the distinctive characteristics of null quantity, comprehensive behavioral and neural research on human and nonhuman animals (e.g., Beran, Perdue, & Evans, 2015; Biro & Matsuzawa, 2001; Howard, Avarguès-Weber, Garcia, Greentree, & Dyer, 2018; Merritt & Brannon, 2013; Merritt, Rugani, & Brannon, 2009; Okuyama, Kuki, & Mushiake, 2015; Pepperberg & Gordon, 2005; Ramirez-Cardenas, Moskaleva, & Nieder, 2016; Zaks-Ohayon et al., 2021a, 2021b) has shown that empty sets can be mapped onto the ANS together with non-empty sets. Specifically, comparisons between empty and non-empty sets result in a distance effect (Moyer & Landauer, 1967) and an end effect (Banks, 1977; Leth-Steensen & Marley, 2000), both of which are considered markers for numeric representation. Accordingly, response latencies decrease with the increase in the numerical value of the non-empty set that is being compared to empty set, and comparisons between empty and non-empty sets are responded to faster than comparisons of non-empty sets, respectively (e.g., Merritt et al., 2009; Merritt & Brannon, 2013; Zaks-Ohayon et al., 2021a, 2021b). Furthermore, single-cell recordings from the ventral parietal and prefrontal cortex of monkeys (*Macaca fuscata* and *Macaca mulatta*) led to identifying two different types of "number neurons" that were selectively activated in response to empty sets. One was an exclusive type, showing increased activity selective to empty sets and decreased activity to non-empty sets, and the other a continuous type, showing maximum activity in response to empty sets and gradually decreased activity to successively larger non-empty sets (Okuyama et al., 2015; Ramirez-Cardenas et al., 2016). Number-selective neurons of the second type were also previously reported for non-null quantities (e.g., Nieder, 2013).

Next, we turn to C&B's question of what kind of numbers can be represented by the ANS. Although we have previously shown that, psychologically, the number 0 can be perceived as a natural number (Pinhas *et al*., 2015; Pinhas & Tzelgov, 2012), it is not considered as such in mathematical terms because it is neither a positive nor a negative integer. However, mathematically, zero can be expressed as the ratio a/b, if it serves as the numerator a, and therefore, fits the definition of a *rational* number, consistent with the type of numbers that are represented by the ANS according to C&B. More generally, when considering what kind (s) of numbers can be represented by the ANS, the inclusion of zero breaks down the orderly relationship that exists between a number's ordinality (i.e., its position in the number sequence) and cardinality (i.e., its numerical value) when only natural numbers are considered (Seife, 2000). Accordingly, if only dealing with natural numbers, 1 is the first, 2 is the second, 3 is the third, and so on. However, when 0 is also included, 0 is the first, 1 is the second, 2 is the third, and so on. Thus, the fact that the ANS represents empty sets as zero indicates that the ordinality and cardinality properties of numbers are no longer interchangeable, and presumably sets the stage for more "complex" forms of numbers to be represented by the ANS.

Clearly, further research is still needed to fully characterize the nature of the representations captured by the ANS. However, by highlighting the case of the empty set, we hope to inspire future research focused on other unique numerical concepts that, similar to zero, may reveal something fundamental about the way numerical information is represented.

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Numerosity, area-osity, object-osity? Oh my

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Abstract

There is ongoing debate about whether number is perceived directly. Clarke and Beck suggest that what plagues this debate is a lack of shared understanding about what it means to perceive number in the first place. I agree. I argue that the perception of number is held to a different standard than, say, the perception of objecthood; considering this, I explore what it might mean for the number system to represent rational numbers.

Forget number for a moment. Consider another question: Do we perceive objects? The answer to this question must be an unambiguous "yes!"; our understanding of attention depends on the notion of "objects."

But what is an object? Let's start with a rectangle. That's obviously an object. If we cut that rectangle right down the middle to form two separate squares, now we have two objects. Simple enough. What about cases in between? Suppose we connect our two separated squares with a thin line. Is this one object or two? If your answer is "two," then ask: How thick does that line have to become before your answer becomes "one"? Now, consider the opposite. Start with a single rectangle, and cut a hole out of the middle of it (and imagine that this hole is as wide as the gap between the two squares you imagined before). Is that still one object? If so, how tall does that hole have to become before the rectangle becomes two objects? These examples hardly scratch the surface of objecthood edge cases.

Both the perception of objecthood and the perception of number blur the lines between the continuous and the discrete. On the one hand, the essence of objects is that they discretize attention; on the other hand, multiple independent cues simultaneously influence our impressions of objecthood (e.g., Feldman, 2007), resulting in continuous effects on attention and on spatial/temporal perception (e.g., Yousif & Scholl, 2019). Similarly, it seems as if number must be discrete (what would it mean to perceive 16.5 objects?); yet, at the same time, we often talk about "numerosity" as if to imply that we perceive something vaguer and more imprecise than a discrete number. The perception of number and the perception of objecthood share this same ambiguity, yet, for some reason, there aren't multiple BBS papers (see also Leibovich, Katzin, Harel, & Henik, 2017) discussing whether we perceive objects – and nobody has yet felt the need to coin the term "objectosity." What's the difference?

Here's my best guess: This boils down to the fact that objecthood has consequences; objecthood influences attention, and that's *measurable* (e.g., Egly, Driver, & Rafal, 1994). If we want to know whether objecthood was manipulated, we can simply ask whether object-based attention effects are attenuated. Imagine though if we did not have measures of object-based attention. How would we determine that objecthood is continuous – having observers make quick key press judgments about which of two things was more "object-like"? We would never be satisfied with such evidence. We may argue endlessly about this confound, or that one. Every few years, someone would come up with a new confound to argue over. We'd devote several BBS papers to discuss this deep, crucial matter of whether the visual system *actually* perceives objects. So it is with number.

How should this influence how we think about number in relation to other spatial properties? Traditionally, we think of number and area as things that *ought* to be perceived separately. There is a general thought that if area interferes with number perception, we must not be perceiving number directly, or veridically (see "The Argument from Confounds"; but see also Yousif & Keil, 2020). Clarke and Beck's argument indirectly raises a radical possibility: If the number system represents fractions, does that mean that the number system represents partial objects (i.e., can distinguish between 16 vs. 16.5 objects)?

If it is true that the number system represents partial objects, it would force us to reconsider how we think about confounds in quantity perception tasks. Consider again this idea that the visual system is tasked with counting *objects*. Now, imagine a display with 20 identical rectangles (see Fig. 1). Suppose we conduct a numerosity estimation task on this display, and we find that observers perceive and represent the display as having approximately 20 things. Now, imagine that we have a similar display, except that seven of the 20 rectangles have been cut in half, such that what remains in the display are 13 full rectangles and 7 half rectangles. Stated differently: there remain 16.5 full rectangles. We should expect that this latter display is perceived as less numerous, and we would traditionally explain this effect in terms of a congruency between number and area. But what if, instead,

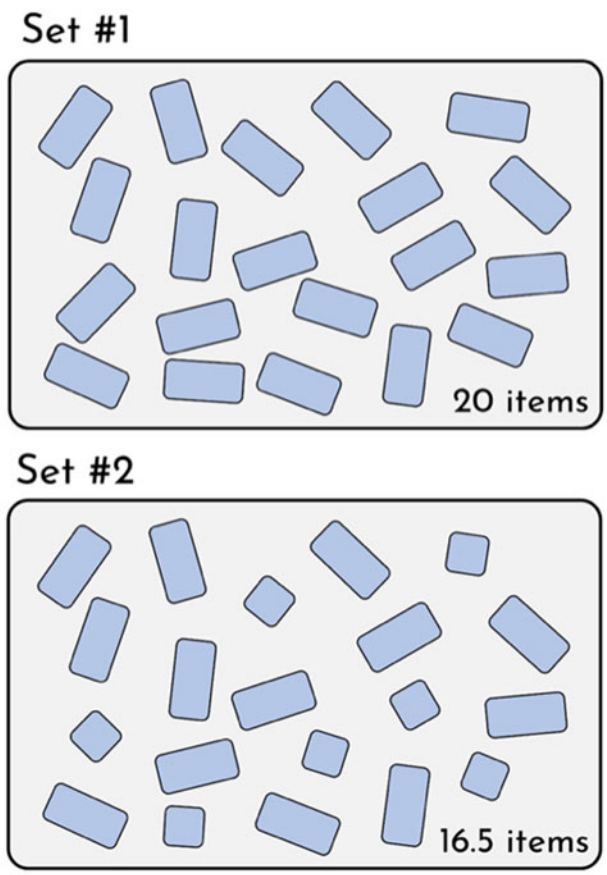

Figure 1 (Yousif). Two sets, one with 20 "full" objects, and one with 16.5 "full" objects. Does the number system represent these partial objects?

the number system is just representing the fact that there are partial entities in the display? What if the number system is representing the rational number 16.5?

If true, this leads to the novel prediction above: that area ought to influence number perception (insofar as area reflects partial wholes). After all, spatial cues influence the perception objecthood (Franconeri, Bemis, & Alvarez, 2009; Yu, Xiao, Bemis, & Franconeri, 2019), and the visual system must be counting objects. This would be ecologically realistic. Seven half meals are equivalent to 3.5 full meals; it could be argued that this latter quantity, and not the former, is the better one to represent. (At the extremes, this would not be viable. One massive object probably ought not to be equated with 100 tiny objects of equivalent size. But, under most circumstances, similar objects are often of a fairly similar size. Edges cases such as these likely would not occur very often in the natural environment. Or, maybe they do! My suggestion is only that future research could consider this intriguing possibility.)

This possibility may or may not pan out empirically, yet it is one of many ways that Clarke and Beck's suggestion could radically alter how we think about quantity perception moving forward.

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Constructing rationals through conjoint measurement of numerator and denominator as approximate integer magnitudes in tradeoff relations

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Abstract

To investigate mechanisms of rational representation, I consider (1) construction of an ordered continuum of psychophysical scale of magnitude of sensation; (2) counting mechanism leading to an approximate numerosity scale for integers; and (3) conjoint measurement structure pitting the denominator against the numerator in tradeoff positions. Number sense of resulting rationals is neither intuitive nor expedient in their manipulation.

The proposal by Clarke and Beck (C&B) that the approximate number system (ANS) represents rational numbers is specific and intriguing. In this comment, I shall speculate on representational mechanisms through which rationals are deployed by the mind as a numerical representation of magnitude continuum.

Representation of magnitude of sensations was long understood, through the ideas of "just-noticeable difference" (Weber fraction) and Fechner's logarithmic scaling of sensation in relation to the intensity of eliciting stimuli. With a series of psychophysical experiments using a variety of direct-scaling paradigms (e.g., magnitude estimation, magnitude production, and cross-modality matching), Stevens (Stevens, 1936, 1956; Stevens & Galanter, 1957) unequivocally established this psychophysical scale as an ordered continuum admitting concatenation operation, and proposed the power-law to map physical intensities to a psychological scale (Stevens, 1957). Fechner's logarithmic scaling and Stevens' power scaling can be reconciled as the former can be taken as the extreme case of the latter, when the exponent β approaches zero:

$$\frac{1}{\beta}(I^{\beta} - 1) \rightarrow \log I.$$

When it comes to numerical representation of the magnitude scale, things become subtler because of co-existing, symbolic and non-symbolic aspects of numerals (Feigenson, Carey, & Spelke, 2002; Feigenson, Dehaene, & Spelke, 2004; Lemer, Dehaene, Spelke, & Cohen, 2003; Xu, Spelke, & Goddard, 2005). Numerosity as the pre-linguistic faculty to represent numerical information, possibly congruent with the psychophysical scale mentioned above, is clearly different from counting (Carey, 2001, 2009; Le Corre & Carey, 2007; Sarnecka & Carey, 2008), which involves the mastery of an array of cognitive routines such as a "successor" function, one-to-one correspondence, and "cardinality" and "equality" of sets for establishing the concept of integer (Carey & Barner, 2019). Whether or not subitization (the process of immediate and accurate recognition of small numbers) plays any role for enabling the counting routine, the resulting system of integers acquires an approximate numerosity scale, and maps linearly or logarithmically to the psychophysical scale (Dehaene, Izard, Spelke, & Pica, 2008; Núñez, Cooperrider, & Wassmann, 2012). This is possible because both counting (which is anchored on "successor function") and numerosity estimation (which is subject to Weber's Law) are predicated on individuation of objects.

The authors referenced the work of He, Zhang, Zhou, and Chen (2009). In that paper, participants were briefly presented with visual displays of dots in random positions and were asked to judge their numerosities; the brevity of stimulus presentations precluded any counting strategies. In some displays, pairs of adjacent dots were connected by line segments whereas in others, line segments were freely hanging without touching the dots (see the reproduced Fig. 1d of C&B). The line segments were introduced to manipulate object individuation aspect of numerosity estimation. Results clearly showed that connecting the pairs of dots by line segments led to an underestimation of dot numbers in those patterns because of decreased individuation. Thus, we suggested "two different stages underlying numerosity perception: first, the individuation of items in a visual display, followed by magnitude estimation based on the distinct number of items" (p. 517).

Numerosity estimation mechanism (approximate integer scale) mentioned above is, of course, congruent with the psychophysical scale revealed in Steven's direct-scaling experiments. In the language of axiomatic foundation of measurement (FoM) (Krantz, Luce, Suppes, & Tversky, 1971), the common underlying measurement structure is that of ordered concatenation group. Their difference is that numerosity scale is countable, and hence the group of integers ($\mathbf{Z}$, +), whereas the psychophysical scale is uncountable, and hence the group of reals ($\mathbf{R}$, +) which is isomorphic to the group of positive reals ($\mathbf{R}_+$, ×). Note that in any ordered concatenation structure (M, ⊕), there is only one binary operator, ⊕, on the set M (which can be $\mathbf{R}$, $\mathbf{R}_+$, or $\mathbf{Z}$) of elements whose magnitude are in total "order." This binary operator ⊕, takes in two elements of the set and outputs another element larger than any of its inputs. That ($\mathbf{R}$, +) and ($\mathbf{R}_+$, ×) are mathematically isomorphic can be easily seen by the identity: $\log(a \times b) = \log a + \log b$. Cohen and Narens (Cohen & Narens, 1979; Narens, 1981) developed the elaborative theory of ratio-scalability for extensive concatenation systems and the associated scale types.

The magnitude system conceptualized as above does *not* automatically come with a multiplication or ⊗ operation that accompanies (and distributes across) the ⊕ operation. That is, the magnitude system is not an algebra with two interwoven operations. To remedy this in FoM, Narens (Luce & Narens, 1987; Narens & Luce, 1986) proposed to use the automorphism group on the representational space, either ($\mathbf{R}$, +) or ($\mathbf{R}_+$, ×), as a surrogate of ⊗. An automorphism σ of a measurement structure is a bijective transformation from the structure to itself (i.e., mapping one element to another) that preserves the order relationship among its elements:

$$\sigma(a \oplus b) = \sigma(a) \oplus \sigma(b), \quad a > b \text{ iff } \sigma(a) > \sigma(b).$$

All automorphisms form (generally non-commutative) group, with group multiplication operation ⊗ implemented as successive application of two automorphisms: $\sigma_2(\sigma_1(a)) = (\sigma_2 \otimes \sigma_1)(a)$. Distributivity of ⊗ over ⊕ always holds, but ⊗ is, in general, a non-commutative multiplication.

In FoM (Krantz et al., 1971), commutative multiplication ⊗ of two elements is through the construction of a conjoint measurement structure. This is the structure involved in tradeoff of Utility and Risk in Value = (Utility, Risk), and Length and Width in Area = (Length, Width). In the current case, we have $\mathbf{Q} = (D, N)$ where D is the denominator and N the numerator of a rational number $\mathbf{Q}$. Here, both (D, +) and (N, +) are integer or numerosity structure ($\mathbf{Z}$, +).

Additive conjoint structure is axiomatized by various cancellation conditions across its two underlying "components." Essential to an additive conjoint measurement is the independence assumptions about its components and the existence of tradeoff, or the "indifference curve" of equal value.

When the rational field $\mathbf{Q} = (D, N)$ is constructed this way, addition and multiplication of two rationals $\mathbf{Q}_1$ and $\mathbf{Q}_2$ can be quite convoluted. Therefore, achieving "dense" representation of magnitude scale using rationals, as C&B suggested, has a heavy price toll when computing the fraction addition and fraction multiplication using only the routines for (D, +) and (N, +). Interference effects should readily be expected.

To summarize, I endorse the proposal of C&B to construct ANS by rationals, and amend it with a suggestion that such representation is achieved through conjoint measurement and tradeoff between the approximate integer (numerosity) representations of the denominator and the numerator.

Conflict of interest. None.

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Authors' Response

Numbers, numerosities, and new directions

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Abstract

In our target article, we argued that the number sense represents natural and rational numbers. Here, we respond to the 26 commentaries we received, highlighting new directions for empirical and theoretical research. We discuss two background assumptions, arguments against the number sense, whether the approximate number system (ANS) represents numbers or numerosities, and why the ANS represents rational (but not irrational) numbers.

We are humbled to have received 26 commentaries from 62 researchers, among them many of our academic heroes. Unsurprisingly, these commentaries reflect a diversity of opinion. Some endorse and build upon the main conclusions of our paper; others highlight points of disagreement. Although we remain confident in our main theses, we learned a great deal from our commentators – about soft spots in our arguments, points that require development, where we could have been clearer, and avenues for future research. We're extremely grateful for their insights.

Our replies follow the order of our target article. We discuss two background assumptions, arguments against the number sense, whether the approximate number system (ANS) represents numbers or numerosities, and why the ANS represents rational (but not irrational) numbers.

R1. Background assumptions

Explanation needs to start somewhere, and our discussion presupposed that the ANS is representational and that it sometimes operates in perception, enabling numbers to enter perceptual contents. Some commentators challenged these background assumptions.

R1.1. Is the ANS representational?

While the idea that the ANS *represents* anything at all is relatively uncontroversial among ANS researchers (but see Beck [2015] for a defense), **Jones, Zahidi, and Hutto (Jones et al.)** suggest that our commitment to representations imports unnecessary "philosophical baggage." They recommend instead embracing an anti-representational *Radical Enactivism*.

In general, we're dubious when people tell us we can avoid philosophical baggage by embracing views with "radical" in the name. **Jones et al.**'s "radical" vision is that we acquire a perceptual sensitivity to numbers simply by virtue of our sensitivity to the affordances they enable: "The 'sevenness' is not a property of the apples, nor of the perceiver, but of what the perceiver can do with them." The trouble is: It's essentially open ended what you can do when you perceive there to be seven of something. So we don't see how the perception of number can be specified in these terms. Furthermore, representation is fundamental to explanations of the ANS's internal computations. For instance, when children use their ANS to add the number of blue dots and red dots in a sequence of events (e.g., Barth et al., 2005), it's not just that they're afforded with (say) the sevenness of the blue dots, the tenness of the red dots, and then magically afforded with the seventeenness of the red and blue dots; they engage in a computational transition, in which internal states of the organism interact in content respecting ways. This presupposes representation.

R1.2. Are numbers perceivable?

Aulet and Lourenco, **Marshall**, **Novaes and dos Santos**, and **Opfer, Samuels, Shapiro, and Snyder** (**Opfer et al.**) all questioned our assumption that numbers are perceivable.

Because numbers are higher-order, **Novaes and dos Santos** and **Marshall** think they cannot be perceived. Marshall proclaims that "second-order entities are no part of the sensible realm," while Novaes and dos Santos write that numbers "emerge only after an agent has adopted a given sortal" and thus are not "out there, inexact or otherwise, to be represented." But we find this puzzling. Surely, it's an objective fact that the apples on the kitchen counter total five in number. This fact is "out there" and does not require anyone's mind to "emerge." It's also the sort of thing one should expect perceptual systems to be capable of picking up on. We're not sure why anyone would think otherwise unless they were committed to an outdated view of perception according to which perception only represents properties for which we have dedicated sensory transducers. But perception is not sensation. At least since Helmholtz, we've known that perception is a constructive and ampliative process whose outputs go beyond its inputs. The visual system makes assumptions that are non-demonstrative but generally ecologically valid. Some of these assumptions concern default sortals that individuate the incoming sensory array into objects and are used to enumerate concrete pluralities. There is no mystery concerning how something higher-order could be perceivable.

Aulet and Lourenco object that our contention that we perceive numbers "conflates the percept with the concept." But studies show adaptation to number in external coordinates (Burr & Ross, 2008; DeSimone et al., 2020), including cross-modal adaptation (Arrighi et al., 2014), which naturally controls for most non-numerical confounds. Because adaptation is a mark of the perceptual (Block, 2014), these studies suggest that we really do *perceive* numbers.

Opfer et al.'s rich commentary questions whether numbers are perceivable on different grounds. Because we assume numbers enter perception through property attribution, they accuse us of assuming the *Identity Thesis*, according to which, "Natural numbers are identical to cardinality properties." They object that the Identity Thesis is linguistically problematic because it's felicitous to say things like (1) but not (2).

(1) The number of women at the party is four.
(2) ?? The number of women at the party is the number four.

We're wary of drawing strong metaphysical conclusions from facts about how people talk (does the language faculty have a hotline to the Forms?) but happily concede that the Identity Thesis might be false. Plausibly, being eight in number is a property, while the number eight is an (abstract) particular. But, contrary to what **Opfer et al.** suggest, we don't think we're committed to denying this or affirming the Identity Thesis. We simply maintain that perception attributes properties like being eight in number, and that the attributed properties are complex, and include reference to natural numbers. For example, the attribution *is eight in number* includes reference to the number eight even though it is not identical to the number eight. (Compare: The property of being as rich as Jeff Bezos is not identical to Jeff Bezos; but when you wish you were as rich as Jeff Bezos you refer to Jeff Bezos.) When we said that "the ANS refers to numbers… by enabling numbers to enter into contents via property attribution," this is what we meant.

Opfer et al. raise a further objection that may now seem pressing: if (as we suggested in our target article) there's a puzzle about how numbers *qua* abstracta could be referred to in perception, and we maintain that attributing cardinality properties involves referring to numbers, how do we avoid our original puzzle? The answer lies in the fact that the puzzle was not supposed to be that it's mysterious how perception could refer to abstract entities. Rather, our worry was that it's mysterious how perception could veridically refer to abstract entities on their own, without simultaneously referring to something concrete. That's why we said, you can't "perceive the number seven itself – on its own." To veridically refer to the number seven in perception, you need to simultaneously perceive a concrete plurality. When you perceive the apples as being seven in number, you refer to a concrete plurality of apples and attribute a property to it, with reference to the number seven occurring within that attribution. (Admittedly, we could have been clearer on this point.)

Now, even though *our* worry didn't concern how perception could refer to abstract objects full stop, this is a worry others might have. But note two things. First, the worry is not unique to perception. Others have worried about how we can *think* about numbers given that they are abstract objects (Benacerraf, 1973). There is, thus, a version of this puzzle that arises for everyone.

Second, with respect to the specific puzzle of *perceiving* numbers, we don't see it as fundamentally different from the puzzle of how one can be perceptually related to other abstracta, such as shapes and colors. Even **Opfer et al.**'s linguistic point doesn't distinguish between these.

(3) The {color/shape} of the ball is {orange/a sphere}.
(4) ?? The {color/shape} of the ball is the {color orange/shape a sphere}.

Admittedly, colors and shapes are often taken to be universals rather than abstract particulars, and if that's right, the two puzzles are not identical. To say exactly *how* they differ, however, would require staking out various controversial positions in metaphysics and the philosophy of mathematics. In an article addressed to an interdisciplinary audience, we tried to bracket such issues. But the two perceptual relations (to shapes/colors and numbers) strike us as sufficiently similar that we've been able to sleep at night. Still, this issue deserves further attention and we're grateful to **Opfer et al.** for highlighting it.

R2. Arguments against the orthodox view

In our target article, we cleared space for the orthodox view that the ANS represents numbers by noting a lack of compelling arguments to the contrary. Thus, our target article replied to three arguments which have been pressed against this orthodox view – the arguments from *congruency*, *confounds*, and *imprecision*.

Notably, few commentators came out in support of these arguments. Indeed, **Marinova, Fedele and Reynvoet (Marinova et al.)** suggest that we "somewhat misinterpreted" the "key message" of the congruency and numerical interference studies they have been involved with – studies which formed the backbone of the arguments from congruency and confounds. Their commentary is helpful because it serves to distinguish two closely related views these studies could be seen to support. A radical interpretation uses them to argue that the ANS fails to exist or represent numbers at all. This radical interpretation seems to be in play when, for instance, Gebuis et al. (2016) claim that congruency studies of this sort indicate that "the output" of the ANS "is not an abstract number" (p. 28). By contrast, a modest interpretation simply takes these results to support an indirect model of ANS processing, on which numerical quantity is derived from continuous percepts. Our aim was to tease these possibilities apart and rebut the radical interpretation. It's good to learn that researchers behind some of these studies also want to distance themselves from the radical interpretation.

Marinova et al. also helpfully observe – and we agree – that it may not be necessary to choose between direct and indirect models of the ANS. Rather, there could be multiple ways perception extracts number, some direct and some indirect.

Aulet and Lourenco were more dubious. They note (correctly) that on our view "whether elephants, mice, or apples" are being counted, the ANS can attribute a numerical value to these "irrespective of their physical differences." Against this, they claim that "number is not (perceptually) independent of other magnitudes," citing evidence that number and area are perceived as integral dimensions. Two dimensions are *integral* when they cannot

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be represented independently and *separable* otherwise. But representing A and B as integral dimensions doesn't preclude representing A. Loudness and pitch are integral dimensions, and each is perceptually represented. Two integral dimensions may even be represented by distinct vehicles, with their integrality (often measured by interference effects) deriving from causal or structural relations among vehicles (Lande, 2020).

Aulet and Lourenco criticize one reason we provided for thinking that number is represented by the ANS. In our target article, we emphasized work by Cicchini et al. (2016) and others, suggesting that when subjects estimate the area, density, and number of dots in a visual array, they are more sensitive to number than area or density, and thus do not simply represent area and density, but also number. Against this, Aulet and Lourenco cite evidence that when perceived area is distinguished from true area, we are less sensitive to number. **Barth and Shusterman** bolster the objection, citing a wider range of studies in support of this claim. Thus, both commentaries suggest that when perceived area is distinguished from mathematical area, it's not true that we're more sensitive to number than area or other non-numerical confounds.

These are fascinating issues. But it's vital to keep three things in mind. First, the correct interpretation of these studies remains hotly contested, so it's probably too early to draw strong conclusions. For instance, Park (in press) objects that studies which control for perceived area tend to introduce *massive* incongruencies between number and non-numerical magnitudes – so, given that incongruencies of this sort suppress numerical sensitivity (DeWind et al., 2015), counterevidence of this sort probably underestimates true numerical sensitivity. Second, our target article did acknowledge some of the counterevidence these authors cite. For instance, we discussed Yousif and Keil's study, suggesting that subjects are more sensitive to "additive area" than number. But, as we noted then, Yousif and Keil are clear that their results cannot be fully explained by non-numerical confounds and still require that numbers are represented. But third, even if none of this were so, the idea that number is uniquely salient was just one reason we gave for rejecting *the arguments from congruency and confounds* (the arguments in which these studies featured). The arguments also fail for independent reasons. For example, the argument from congruency overgeneralizes in absurd ways; and the argument from confounds relies on an *ad hoc* strategy of explaining away success in number tasks that struggles to explain key findings (e.g., cross-modal comparisons and dumbbell effects).

R3. Number versus numerosity

We next proposed that the ANS represents numbers rather than numerosities or other exotic entities. To this end, we observed that the ANS tracks the cardinal number of entities in concrete pluralities (albeit imprecisely), supports arithmetic computations, and exhibits a higher-order sensitivity that's characteristic of number representation. We also argued that the thesis that the ANS represents number admits of no plausible alternatives, promotes integration with other sciences, and avoids a curious double standard with respect to the treatment of non-numerical quantities. Our commentators pushed back on many of these claims.

R3.1. Higher-order sensitivity

One reason to think the ANS represents number is that the ANS is sensitive to the higher-order character of number. Numerical quantities are assigned relative to a sortal, and this distinguishes them from other kinds of quantity. To illustrate, note that the group entering the restaurant is one party of diners, four couples, and eight people, while the group's weight remains constant irrespective of which sortal we apply. That the ANS is sensitive to this higher-order feature of number is especially clear from the dumbbell studies reviewed in our target article (Franconeri et al., 2009; He et al., 2009). Judgments of the number of items are influenced by whether they are connected to one another (even though subjects are told to ignore the connecting lines), suggesting that the system takes a stand on how items are individuated.

Marshall complains that we "hang *an awful lot*" on these dumbbell studies in making this point, but he doesn't question our logic or criticize the studies themselves. By contrast, **Buijsman** objects that the dumbbell studies only contain "a relatively small number of connected dots/squares" and that, as such, performance might result from the object-tracking (or subitizing) system rather than the ANS. But this worry appears to be based on a misinterpretation of the original studies. Buijsman writes, "the fourth experiment of Franconeri et al. (2009) has four circles, and in the connected format these form two dumbbell shapes." While that accurately describes the *figure* accompanying Franconeri et al.'s fourth experiment, the text clarifies that the actual stimuli consisted of 12, 24, or 48 circles, of which 0, 25, 50, 75, or 100% were connected.

This is not a one-off finding. Fornaciai et al. (2016) report that numerical adaptation effects are influenced by whether the post-adaptation stimuli consist of 20 unconnected dots or 10 pairs of connected dots. Fornaciai and Park (2018) confirmed that displays of 16 or 32 dots were underestimated when they were connected (compared to displays containing unconnected dots). In fact, the stimuli needn't really be connected. Kirjakovski and Matsumoto (2016) found that pacman-like stimuli that only *appeared* to be connected via Kanisza-like illusory contours also caused subjects to underestimate their total.

Aulet and Lourenco object that the dumbbell studies do not reveal that the ANS has a higher-order character because "if number perception was genuinely second-order, then it should be just as easy to continue perceiving the number of dots, instead of being biased towards the number of dumbbells." But this worry conflates two things: whether the ANS is higher-order, and whether the sortals it uses are under voluntary control. Crucially, the ANS could be higher-order even if the sortals it uses aren't under voluntary control.

Consider that the visual system is biased toward individuating the world into what are sometimes called *Spelke objects* – bounded, coherent, three-dimensional, continuous wholes (Carey, 2009; Spelke, 1990; but see Green, 2018.) Consequently, when the ANS takes inputs from the visual system, it enumerates Spelke objects by default. That is what the dumbbell studies show because connecting two items turns them into a single Spelke object. These studies evince a higher-order character to the referents of ANS representations because they show that the ANS is applying a sortal – the sortal *Spelke object*. This default can be overridden to some extent (subjects do not treat 10 pairs of connected circles as numerically identical to 10 unconnected circles), but not completely. Moreover, it is only the default in certain circumstances. The ANS also spontaneously enumerates events such as rabbit jumps, heard tones, and, as **Burr, Anobile, Castaldi, and Arrighi** demonstrate, self-generated actions such as hand taps. Thus, the sortal used by the ANS *is* capable of varying, even if (like most of the mind) it isn't under full voluntary control.

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R3.2. Numerosity

In our target article, we objected to the idea that the ANS represents *numerosities* rather than *numbers* themselves. For one, we objected that, while the term "numerosity" is widely used, no one seems to know what a numerosity is. This prompted many commentators to tell us just *what* a numerosity is, although their disagreements are notable, suggesting that insofar as researchers associate a distinctive meaning with "numerosity," it's not universally shared.

In their commentaries, **Núñez, d'Errico, Gray, and Bender (Núñez et al.)** and **Novaes and dos Santos** trace the term "numerosity" to S. S. Stevens. As Núñez et al. report, Stevens (1939/2006, p. 23) defined numerosity as "a property defined by certain operations performed upon groups of objects." We'll confess, this doesn't exactly clear things up for us.

Some commentators had more to say. **Novaes and dos Santos**, along with **Bermúdez** and **Opfer et al.**, suggest that numerosities are *cardinalities*. By contrast, **Núñez et al.** double down on the idea that the ANS is *quantical* rather than numerical, while **Buijsman** and **Gross, Kowalsky, and Burge (Gross et al.)** defend the view that numerosities are *pure magnitudes*. We discuss each of these proposals in the following sub-sections. First, however, we want to reply to two further commentaries that defend the concept of numerosity without neatly fitting into these three proposals.

Barth and Shusterman wonder whether researchers share an "understanding of what 'numerosity' means." Given the variety of proposals made by our commentators, we think it's clear they do not. The common term masks a diversity of concepts. Still, Barth and Shusterman think that the term should be retained. According to them, "number" is ambiguous because it can refer to a number word, a numeral, a mathematical entity, or a property of a stimulus. They think it is useful to have a term that refers just to the last of these, and that "numerosity" is up to the task.

Our view is that words and numerals are clearly not numbers – no more than the word "square" has four equal sides or the dinner bell is fit to eat. Therefore, we don't think anyone should be concerned about confusion on that front. (If one is concerned, using "number word" and "numeral" for number words and numerals, respectively, should guard against mix-ups.) We do think there's a difference between mathematical entities and properties of stimuli, but that's not a distinction that's unique to numbers. When a mathematician says, "A square is a plane figure with four equal sides and right angles," she's talking about a mathematical entity, not a property of a stimulus. But the tiles on Rachael Ray's kitchen floor can have the property of being square just as surely as they can have the property of being 30 in number. Would **Barth and Shusterman** also want to introduce the term "squareosity" to capture the shape property that these stimuli can have? If not, why introduce "numerosity" to capture their numerical property? Barth and Shusterman don't say.

Gallistel also defends the term "numerosity," arguing that "coherent discussion" requires a three-way distinction between *numerons*, *numbers*, and *numerosities*. A numeron is "a symbol in a computing machine like the brain." This strikes us as a helpful concept. Just as we use *numerals* (e.g., in Arabic notation) in language, the brain uses numerons in its internal code. Numerons are thus vehicles of representation – symbols in the language of thought. But Gallistel also says that *numbers* are symbols. This leaves us confused. If numerons and numbers are both symbols, aren't they the same thing? And wouldn't numbers then be vehicles of representation too? This seems like a mistake, analogous to confusing a rose with the word "rose." Symbols refer to numbers, but they aren't identical to numbers. After all, different symbols can refer to the same number (e.g., "4," "four," and "IV"), and the same symbol can refer to different numbers in different notations (e.g., "100" refers to one hundred in decimal notation and to four in binary).

Finally, **Gallistel** claims that a *numerosity* is "the number you get when you correctly count" a collection. But, if a numerosity is just a number, Gallistel has one more distinction than he needs.

Gallistel provides one further reason to think that we need "numerosity" in addition to "number." Just as psychophysicists use "brightness" for the percept and "luminance" for the distal stimulus, they need "number" for the percept and "numerosity" for the distal stimulus. But, while some objective magnitudes have an associated term that naturally applies to the percept (e.g., luminance/brightness, sound wave amplitude/loudness, and sucrose concentration/sweetness), others do not (e.g., distance/?, duration/?, and area/?). And yet psychophysicists seem to get along just fine measuring these percepts. As such, we should ask what feature of the percept "numerosity" is supposed to capture. Is it the vehicle? Gallistel already gave us "numeron" for that. Perhaps, instead, it's the phenomenal character of the percept (as it might be with "brightness")? But, beyond the oddity of using "number" to refer to a phenomenal property, it's far from clear that there's a phenomenal character that's common to how number is represented in vision, audition, action, and so on. The distinction between number and numerosity serves no apparent purpose.

R3.3. Cardinalities

Novaes and dos Santos write that "in the contemporary literature one finds 'numerosity' defined as a synonym for cardinality." In defense of this claim, they cite Nieder (2016, p. 366), who writes, "Cardinality (also known as numerosity) corresponds to the empirical property of quantity, and is the number of countable elements in a given group (for example, five runners)."

On one interpretation of this passage, cardinalities are just a specific type of number: cardinal numbers. And to represent a cardinality is to represent a cardinal number. This interpretation is obviously consistent with the hypothesis that the ANS represents numbers.

On a second interpretation, cardinalities are properties of concrete pluralities rather than numbers themselves. This would put **Novaes and dos Santos** in line with **Barth and Shusterman** and **Opfer et al.** Here, a distinction is drawn between the number five (a mathematical entity) and being five in number (a property of a concrete pluralities). To represent the cardinality of the runners is to represent the runners as being five in number – that is, as having a particular property. As we noted in section R1.2, we agree that the ANS attributes cardinality properties in this sense; but we maintain that in so doing it refers to numbers. Therefore, this proposal is also compatible with our hypothesis.

Is there some other way to use cardinality as an alternative to number? The notion of a cardinality derives from set theory, and **Novaes and dos Santos** suggest appealing to the set-theoretic notion of one-to-one correspondence, such that two sets have the same cardinality if and only if their members can be put in one-to-one correspondence. **Bermúdez** develops this suggestion, showing how it predicts that the ANS can represent *comparative*

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properties, but not *absolute* properties. That's because a computation of one-to-one correspondence can tell you whether two collections are equinumerous or not, but not how many elements are in either one. Bermúdez argues this proposal is consistent with much of the data associated with the ANS, including the many studies that require subjects to determine which of two presented pluralities is greater, and Weber's Law itself.

One might worry that this proposal falsely predicts that numerical comparisons will be precise, because the operation of one-to-one correspondence is precise. (It grounds some definitions of the integers.) Carey and Barner (2019) reject the proposal on exactly these grounds, writing that "the ANS lacks a mechanism like one-to-one correspondence that can establish the exact equality of sets" (pp. 826–827). But we see no reason that noise couldn't corrupt a computation that places representations in one-to-one correspondence, thereby giving rise to the imprecision associated with the ANS.

There are, however, two difficulties with the proposal. First, while numerous studies show that ANS representations can be stored in working memory, working memory for seen objects degrades quickly after three or four objects (Alvarez & Cavanagh, 2004; Vogel et al., 2001). A large collection of objects is not represented in memory in full detail, but as an "ensemble" using summary statistics (Alvarez, 2011). For example, the mean area of a collection of dots might be recorded in memory, but not the individual areas (Ariely, 2001). And similarly for orientation, brightness, location, and other properties. But, if the collection of individual objects in a display aren't stored in memory, then the comparative cardinality view cannot explain how subjects perform numerical comparisons once that collection is no longer perceivable. Consider the studies by Barth et al. (2005) that **Bermúdez** cites. In the very first experiment, preschoolers see some dots on a screen, then see those dots being covered, and then see some new uncovered dots. They then have to say whether the covered dots are more or less numerous than the uncovered dots. To do this, they must maintain in memory either a representation of the covered dots themselves or a summary representation of the covered dots' number. If they maintained a representation of the covered dots themselves, then they could put those dots in one-to-one correspondence with the still-visible dots to determine their comparative cardinality. But the displays contained up to 58 dots, well above the limits of visual working memory. Therefore, memory must instead store a summary representation of their total number.

Second, when **Bermúdez** writes, "Clarke and Beck readily concede that there is no evidence that the ANS is sensitive to the successor function or to basic arithmetical operations," he's only half right. We did concede that the ANS isn't sensitive to the successor function. But we noted "that ANS representations enter into arithmetic computations such as greater-than and less-than comparisons, addition, subtraction, multiplication, and division." This matters because most arithmetic computations require more than one-to-one correspondence. While other set-theoretic operations might be appealed to (e.g., addition might be explained in terms of the union operation), this approach gets trickier when we consider that ANS representations are believed to enter into arithmetic computations with other magnitudes. For example, there is evidence that the mind takes representations of number and divides by its representations of duration to yield representations of rate (Gallistel, 1990). We find it hard to envision how comparative cardinalities can explain such computations. (**Núñez et al.** claim it's a "biological no-go" to suppose that the nervous system implements arithmetic operations such as division. But they don't explain why; nor do they provide an alternative explanation of the many studies we cited that are indicative of such operations.)

R3.4. The quantical

Núñez et al. accuse us of "biological misconceptions," "mathematical naïveté," "serious inconsistencies," having "only [one] novel claim," "erroneous" characterizations, "misrepresent[ing]" distinctions, "an unnecessary condescending tone," and torturing puppies for fun. (We're reading between the lines on that last one.)

In an earlier article, Núñez argued that the capacities associated with the ANS "are not about numbers, but are about quantity, and therefore should not qualify as numerical… I propose to refer to these biologically endowed capacities as *quantical*" (Núñez, 2017, p. 419; emphasis in original). We interpreted these claims as implying that the ANS does not represent number, and instead represents something "quantical." **Núñez et al.** stress that "quantical" is an adjective to describe non-numerical quantities, and not a noun as we sometimes used it in our article. Fair enough. But, if the ANS is about something "quantical" rather than something numerical, what *exactly* does it represent? Núñez (2017) tells us that "quantical" pertains to quantity. But as we stressed in section 5.3 of our target article, just saying that the ANS represents quantities doesn't capture its second-order sensitivity or distinguish it from systems that represent magnitudes such as distance or duration.

Núñez et al. offer some clarificatory remarks. For one, they say that the quantical–numerical distinction is not about (im)precision. This was one of Núñez's (2017) stated reasons for thinking that the ANS is non-numerical, when he wrote, "A basic competence involving, say, the number 'eight,' should require that the quantity is treated as being categorically different from 'seven,' and not merely treated as often – or highly likely to be – different from it" (p. 417). And again, when he wrote that quantifying "in an exact and discrete manner" is part of the "minimal criteria" for a capacity to be numerical (p. 418). In section 5.3 of our target article we argued that this is not a good reason to reject the hypothesis that the ANS represents numbers. Núñez et al. seem to agree.

Núñez et al. claim that the core difference between quantical and numerical cognition lies in the distinction between non-symbolic and symbolic reference. By a "symbol" they seem to mean public symbols from a spoken or written language, and not internal mental symbols such as **Gallistel**'s numerons. (Thus, they deny not only that the ANS is symbolic, but also that subitizing is symbolic even though subitizing has been argued to recruit demonstrative-like mental symbols [Pylyshyn, 2007].) The way they use the quantical–numerical distinction is open to two interpretations, however, one weak and one strong.

According to the weak interpretation, the distinction is merely supposed to emphasize that the capacities that come online with public numerical symbols are importantly different from the capacities associated with the ANS. We agree wholeheartedly and said as much in our target article. Mastering a public numerical system makes it possible to do things that one could not do before. According to the strong interpretation, not only are the capacities different, but also the capacities associated with the ANS *are not numerical*, and so the ANS does not represent numbers. For reasons glossed above, we interpret Núñez (2017) as endorsing this stronger interpretation. But, as we note in our

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target article, the stronger claim faces two problems **Núñez et al.** don't address. First, it struggles to explain the higher-order sensitivity of the ANS. And second, it owes an account of what the ANS represents, if not numbers. Saying that it is "quantical" is insufficient because that either reduces to the trivial claim that internal mental representations are not public symbols (if "quantical" is just taken to mean *not symbolic*) or else fails to distinguish the capacities associated with the ANS from the capacities associated with systems devoted to quantities like duration or distance (if "quantical" means *quantitative*).

R3.5. Pure magnitudes

Most researchers who claim that the ANS represents numerosities fail to adequately explain what a numerosity is. Burge (2010) is an exception. His proposal that the ANS represents Eudoxan pure magnitudes is substantive and specific. We view this as the leading competitor to our proposal that the ANS represents number. **Buijsman** and **Gross et al.** defend it.

Gross et al. argue that pure magnitudes are preferable to natural numbers for two reasons. First, the ANS isn't sensitive to the full structure of the natural numbers. For example, its capacities do not include "counting, one-to-one matching, or a successor operation." By contrast, they claim pure magnitudes have all the structure needed to explain the ANS, and no more. We disagree. Pure magnitudes are extremely fine grained. The ancient Greeks introduced them to capture ratios that we would now express using irrational numbers. But, as we argued, there is no evidence that the ANS is sensitive to irrational numbers. Pure magnitudes have *more* structure than is reflected in the ANS (Beck, 2015).

Second, **Gross et al.** argue that perception already represents pure magnitudes when it represents continuous magnitudes like length and weight. To motivate this claim, they appeal to Peacocke's (1986) thesis that perception is unit-free. (When you see the length of a piano, you don't represent that length in meters, yards, or any other units.) But, given a background realism about magnitudes like length, the view that perception represents these can also respect the unit-free character of perception. Veridical perception of length is unit-free because length itself is unit-free (Peacocke, 2020). Pure magnitudes aren't needed. Furthermore, even if pure magnitudes were needed to represent continuous magnitudes, it wouldn't follow that they are also needed for the ANS unless continuous magnitude representations and the ANS draw on the same representational elements. While this hypothesis has been defended (Feigenson, 2007; Walsh, 2003), some recent evidence speaks against it. For example, Odic (2018) found that the precision of continuous and numerical magnitude representations follows distinct developmental trajectories.

We argued that the ANS represents numbers rather than pure magnitudes because only numbers have a second-order character and the ANS exhibits sensitivity to a second-order property of collections. **Buijsman** thinks the pure magnitude hypothesis "cannot (yet) be dismissed" because he is skeptical that ANS representations are genuinely second order. But, as we explain above (section R3.1), these concerns are misplaced.

Gross et al. grant that the ANS exhibits second-order sensitivity, but claim that this is equally well captured by the pure magnitude hypothesis. On their view, perception represents a variety of magnitude types in terms of pure magnitudes, including distance, weight, duration, and "aggregate membership." (Whereas sets are abstract, an *aggregate* is roughly what we called a "concrete plurality," or a spatiotemporally located collection.) When pure magnitudes measure continuous magnitudes like distance, weight, and duration, sortals are not involved. But, when they measure aggregate membership, sortals must be involved. Thus, Gross et al. conclude that representations of pure magnitudes can also exhibit second-order sensitivity.

We're not so sure. To see why, it's helpful to distinguish the *genus* pure magnitude, which divides "into discrete and continuous subspecies" and "is not specific to any further type of magnitude – such as spatial extent or size, temporal duration, weight, and so forth" (Burge, 2010, p. 482), from various *species* of pure magnitude, such as duration and weight. We interpreted Burge (2010) as claiming that the ANS represents the genus pure magnitude. On that interpretation, we think our original criticism stands. The genus pure magnitude does not differentiate between being first order or second order; but the ANS is second order; so ANS representations are, in that respect, not well captured by the genus.

By contrast, **Gross et al.** seem to take the ANS to represent a particular species of pure magnitude – namely, the discrete species that measures "aggregate membership" (see also **Ball**). This evades our criticism because the discrete species is second order. But, as we understand it, this discrete species of pure magnitude just is natural number. What makes numbers a species of pure magnitude is that they can stand in ratios (analyzed in terms of equimultiples). But the ancient Greeks held that there is more to numbers than that. For example, they maintained that numbers are composed from discrete units. While it's true that they didn't attempt a reductive definition of number in terms of the successor relation or one-to-one correspondence (that would have to wait for the late nineteenth century), it doesn't follow that they were talking about something else. Therefore, if Gross et al. take the ANS to represent the discrete species of pure magnitude that measures aggregate measurement, that sounds to us like another way of saying that the ANS represents natural numbers.

R3.6. The scientific-ontology bias

We argued that entities that appear in our scientific ontology should be favored as contents of the ANS. We agree with **Gross et al.** that this consideration is only *prima facie*. It can be overruled by other considerations. While we take it to be an advantage of our hypothesis that it meets this consideration, we never meant to claim the advantage as unique. Other hypotheses may meet it too.

Brown objects that psychologists justifiably attribute contents that are not part of our scientific ontology. For example, developmental psychologists attribute representations that do not differentiate between heat and temperature. But, in that case, there is overwhelming evidence that children systematically conflate heat and temperature, so neither content on its own is appropriate. The consideration we adduced is thus overruled. But we argued at length that the ANS does not systematically conflate number with other magnitudes.

Brown also considers color vision. If we say that the ANS represents numbers, shouldn't we also say that color vision represents wavelengths? We think not. The way the ANS represents number (and, for that matter, the way perception represents distance, duration, weight, and a host of other magnitudes) is lawlike. By contrast, the relation between wavelength and color percepts is notoriously arbitrary.

If a bias toward scientific ontology can be overruled, is it needed? One reason to think so is that mental representation is always noisy and imprecise. The mind is an imperfect instrument

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with limited resources. Thus, one can always improve the fit of a content assignment by inventing a new concept that accommodates the noise and imprecision. But that would lead to overfitting, idiosyncratic contents, and a missed opportunity to capture genuine connections between the mind and world. The bias serves a useful purpose.

R3.7. Modes of presentation

In claiming that the ANS represents number, we did not mean to deny differences between ANS representations and mature number concepts. We simply argued that these could be captured by differences in their modes of presentation. Such differences are important, and we certainly didn't mean to treat them as "an afterthought" (**Barth & Shusterman**). On the contrary, we emphasized differences in mode of presentation.

Jones et al. think our appeal to modes of presentation is problematic. As they see it, "the distinction between the 'sense' and 'reference' of neural representations is an ad hoc construction without any independent justification." They grant that modes of presentation are legitimate when applied to person-level states, like experiences or beliefs, but deny that this is so when applied to sub-personal representations.

We disagree. For one, sub-personal states can differ in format (Marr, 1982), and this implies differences in mode of presentation because computational work is needed to translate between format types. Elsewhere, both of us have argued that ANS representations differ from conceptual thoughts in precisely this way (Beck, 2015; Clarke, forthcoming). But ANS representations are also not purely sub-personal. When you perceive a group of dots, they look to be a certain number to *you* and not just some component of your brain/mental machinery (see the demonstration in Burr and Ross, 2008).

Peacocke helpfully characterizes ANS representations as having the form *roughly that many Fs* and teases apart three aspects of their modes of presentation. ANS representations are *unspecific* (they don't refer to one specific number), *non-canonical* (they don't use a canonical system of representation to refer to numbers), and *non-recognitional* (the ANS doesn't enable subjects to reliably recognize the same number presented at two different times). Peacocke suggests that these three features should be distinguished conceptually because they co-occur only contingently. There could be a numerical perceptual system that was specific (unlike the ANS) but also non-canonical and non-recognitional. Its mode of presentation would have the form *that many Fs*. It's unclear to us, however, what would ground the specificity of this hypothetical system. If you say, "That many Fs," the specificity plausibly derives from your mature counting abilities, or at least an ability to place items in one-to-one correspondence. In communities lacking those abilities, an utterance of "That many Fs" would not be specific. By contrast, if we are meant to imagine that the specificity is grounded in the perceptual system's discriminative abilities (in the way that having perfect pitch grounds reference to a specific pitch in someone who says "that pitch"), then the system is plausibly recognitional too. While this doesn't show that it's *impossible* for being specific, recognitional, and canonical to come apart in the ways Peacocke suggests, there may be important and deep connections among them.

R4. What kind(s) of number?

The preceding discussion notwithstanding, many commentators sympathize with our suggestion that the ANS represents numbers. But our target article considered a further question: What *kinds* of number does it represent? We speculated that the system goes beyond representing natural numbers by representing rational numbers. At the same time, we expressed skepticism that the ANS goes so far as representing irrational numbers and, hence, the reals more generally. Various commentaries pick up on these claims.

R4.1. Rational numbers

Some commentators welcome our suggestion that the ANS represents rational numbers. **Libertus, Duong, Fox, Elliott, McGregor, Ribner, and Silver** highlight evidence that ANS acuity predicts math skills at school, and suggest that it may be fruitful to explore whether the ANS's involvement in rational number processing relates to children's later understanding of fractions and decimals. This could be an important application of the conjecture.

In a similarly constructive spirit, **Zhang** draws on measurement theory to offer a technical proposal for how rational numbers might be constructed from placing ANS representations in the numerator and denominator of a fraction. Meanwhile, **Yousif** notes that the ANS's (alleged) representation of rational numbers offers to reframe findings typically interpreted as congruency effects. A bias toward treating smaller objects as fewer may result not from congruency effects of area/volume on number, but from interpreting the smaller objects as partial objects. This possibility is certainly worth testing. We also agree with his suggestion that more attention should be paid to the concept of a visual/perceptual object (Green, 2018, 2019).

Extending our conjecture, **Pinhas, Zaks-Ohayon, and Tzelgov** review fascinating evidence that the ANS represents zero. If that's right, we should say that the ANS most basically represents non-negative integers (0, 1, 2, …) rather than natural numbers. But, if zero enters into rational number representations in the way positive integers might, an intriguing possibility arises: Zero might feature as the denominator of a fraction, enabling the ANS to represent infinity!

Ball, **Gómez**, **Lyons**, and **Peacocke** took a more skeptical tone. In our target article, we proposed that, while the ANS most basically represents natural numbers, those natural numbers can enter into ratios, indicating that the system represents rational numbers as well. But these commentators worry that representing a ratio of natural numbers is not the same as representing a rational number.

As **Peacocke** puts the concern: "Appreciation that 6:4, 12:8, 3:2 are all the same ratio is not yet encoding that ratio as a single number 1 ⅔." He then proceeds to claim that the evidence we cited in favor of our proposal only supports the conjecture that the ANS represents ratios, not rational numbers as we suggest. But what's required for representing something as a rational number as opposed to a *mere* ratio? At one point, Peacocke says that this would involve representing these "as having a certain position in a rational number line." This suggests that if the ANS could go beyond matching ratios (e.g., representing that 6:4 and 12:8 are equal) by ordering these into greater/lesser relations (e.g., representing that 6:4 < 7:4), then this would go some way toward showing that the ANS is capable of representing rational numbers. But, if this were all that's required, our proposal would be favored by the studies we described in which subjects use their ANS to gamble on the more favorable of two ratios (Matthews & Chesney, 2015; Szkudlarek & Brannon, 2021).

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Lyons thinks the evidence we cited fails to show that rational numbers are represented because it fails to show that the ANS represents non-natural rational numbers greater than 1. But there are non-natural rational numbers less than 1 (e.g., ½). So, even if Lyons were right, the ANS could still represent *some* non-natural rational numbers. Moreover, the fact that subjects can gamble on the more favorable of two ratios suggests that they can distinguish a ratio of (say) 2:3 from a ratio of 3:2, and thus, if they represent any rational numbers at all, they do not merely represent non-natural rational numbers less than 1.

Gómez notes that distance effects (a signature of the ANS) show up when subjects compare certain symbolic numerical representations (e.g., single Arabic digits), but appear less consistently when they compare symbolic fraction representations. This leads him to infer that the ANS may not represent rational numbers. But, whether the ANS represents rational numbers is one thing; whether it maps those representations onto symbolic fractions is another.

Ball offers a means of adjudicating whether the ANS represents rational numbers or just ratios. He notes that extensive magnitudes (numerical or otherwise) can be added to one another, while intensive magnitudes cannot. But rational numbers are extensive (½ and ¼ can be summed) while ratios are not (1:2 and 1:4 cannot be summed). To decide whether the ANS represents ratios or rational numbers, we should thus investigate whether the ANS can *add* rational numbers.

In short, we *love* this suggestion. While we don't know of existing evidence that speaks directly to **Ball**'s point, it nicely distinguishes ratios from rational numbers and is empirically testable. (See footnote 6 of our target article for a complementary suggestion about how to distinguish ratios from fractions.)

R4.2. Precision

Lyons takes the ANS's imprecision to imply that it represents "approximate number" (e.g., 13ish), suggests that this is at odds with our proposal, and claims that this is something which cannot be "easily squared" with our suggestion that the ANS represents rational numbers – a conjecture which attributed "greater precision [to the ANS]… when what was needed was less." But we suggested that the ANS might represent "numerical intervals (5–9, 1.25–1.75, etc.) (Ball, 2017), or probability distributions over numerical intervals." Either option would involve the ANS referencing numbers, and be compatible with the representation of rational numbers. If Lyons has something else in mind by "approximate number" and "13ish," it's not clear to us what it is.

Lyons also claims that ANS imprecision should be attributed to ANS content, and not ANS vehicles. This leads naturally to the view that the ANS represents a (probability distribution over) a range of values. When you see 10 dots flashed on a screen, you represent there being 8 to 12 dots (or a bell-shaped probability distribution that peaks at 10). But that can't fully capture the imprecision in the ANS. For if it did, then when queried as to the number of dots, you should be able to reliably report the midpoint of the range or the peak of the distribution (i.e., 10). But subjects cannot do that. Some of the imprecision associated with the ANS is exogenous to its content.

R4.3. Is the RPS part of the ANS?

Commentators such as **Dramkin and Odic**, **Hecht, Mills, Shin, and Phillips (Hecht et al.)**, and **Hubbard and Matthews** raise a quite different worry for our hypothesis. They concede that rational numbers are represented but deny that the ANS *itself* produces these representations. Rather, they think that there is a separate domain-general ratio processing system (RPS) that does all the computing over ratios (for numbers, durations, distances, etc.).

It's important to recognize that this algorithmic-level hypothesis is consistent with our computational-level hypothesis that the ANS represents ratios. The key to seeing this is noting that "ANS" is ambiguous. We use it to refer to a system that is individuated in terms of its function: representing and computing over numbers in accordance with Weber's Law. But these commentators use it to refer to a module that's individuated by its inputs, outputs, and algorithms. On their proposal, what we call the ANS is realized by (at least) two modules: a number-specific module (which confusingly is also sometimes called the ANS) and a domain-general module for processing ratios (the RPS). Of course, the RPS could be a component of non-numerical systems too. (The respiratory system is distinct from the circulatory system, but the lungs belong to both).

On our usage, the ANS is distinctive because it concerns numbers (rather than other magnitudes) *and* because it obeys Weber's Law (unlike other numerical or quasi-numerical systems, such as the subitizing system). Our conjecture was thus not wedded to any given account of the system's underlying architecture. By analogy, we noted that the visual system is often said to be unified by its computational level description, despite comprising myriad submodules (Clarke, 2021; Marr, 1982).

Henik, Salti, Avitan, Oz-Cohen, Shilat, and Sokolowski acknowledge the point about levels of description but reject the proposed unity of the ANS, claiming that neurophysiological evidence supports a multi-system architecture which involves at least one generalized magnitude system (cf. Walsh, 2003). But, even bracketing evidence that tells against a generalized magnitude system (Odic, 2018; Pitt et al., 2021), such possibilities are precisely what a computational level description of the system leaves open (Marr, 1982).

In emphasizing a computational level description of the ANS we didn't mean to suggest that an algorithmic or neurophysiological description of the system is unimportant. Indeed, our target article offered some brief speculations on this point. For instance, we tentatively suggested that the ANS's representation of rational numbers may derive from its first assigning natural numbers to concrete pluralities and only then deriving ratios or rational numbers from the relations between these. In so doing, our speculations went beyond a bare computational level description, suggesting possible stages of processing in the ANS's analysis of rational numbers. These speculations were put under pressure by **Hubbard and Matthews**. They noted evidence that one's capacity to discriminate ratios is not correlated with one's acuity discriminating natural numbers under relevant conditions, and that ANS training does not transfer to ratio tasks. Insofar, as these studies are successfully measuring *numerical* ratios, they are hard to square with our tentative proposal. Note, however, that they are also hard to square with the proposal by **Hecht et al.** and **Dramkin and Odic** that there is a domain-general RPS that takes inputs from a variety of magnitude-specific modules. For that proposal also predicts that ANS acuity and numerical ratio acuity should be correlated. In any case, we agree with Hubbard and Matthews that "More research is necessary for the final adjudication" and look forward to learning about future findings in this area.

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R4.4. Irrational numbers

In proposing that the ANS represents rational numbers, we stopped short of claiming that it represents irrational numbers and, hence, the reals more generally. **Gallistel** now agrees. But, while we take it to be a contingent matter that the ANS cannot represent irrational numbers, Gallistel thinks this nomologically necessary, claiming that no irrational number "can be represented exactly by any physically realized system."

We're reluctant to go this far. The symbols "π" and "$\sqrt{2}$" represent exact irrational numbers. Perhaps, **Gallistel** simply means that use of the symbol by a physical system will never be perfectly precise. But, short of assuming the sensitivity principle, which we were at pains to reject (and which no commentaries sought to defend), it's hard to see why this should rule out the representation of irrational numbers.

Our proposal was that extant behavioral evidence fails to support the suggestion that the ANS represents irrationals. In saying this, we acknowledged that future research could, potentially, uncover evidence in favor of this suggestion. For instance, if behavioral evidence were to suggest that the ANS is involved in calculating square roots, this might provide evidence that we had not gone far enough.

Dramkin and Odic object to our emphasis on behavioral studies. They point out that behavioral measures can struggle to disambiguate performance from competence and may, therefore, lead us to underestimate the full range of numbers the ANS represents. This is a genuine methodological worry. But, to overcome these limitations, Dramkin and Odic claim that emphasis should be diverted away from behavioral evidence and instead placed on psychophysical models of ANS performance which treat "perceptual signals as highly continuous and in the domain of the reals."

While we don't wish to downplay the importance of psychophysical models, we're not convinced. The potential problems are two-fold. First, models are always idealizations (Weisberg, 2013). They allow us to abstract away from details of the real world, and it's not always clear whether details of the model reflect simplifying assumptions or not. A good model answers not only to reality, but also to the convenience of the modeler. Thus, the "highly continuous" signals in models may not reflect psychological reality. Second, it's important to distinguish the question of whether a model posits internal signals that are continuous from the question of whether the model posits representational contents that are continuous. A continuous vehicle can represent discrete contents. Thus, even if the models to which **Dramkin and Odic** allude were committed to a continuous perceptual signal, it wouldn't follow that they were committed to continuous contents.

R5. Concluding remarks

Our defense of the view that the ANS represents number, and our attempts to clarify the kinds of number it represents, have divided opinion. While we remain optimistic about the main proposals in our target article, understanding what the ANS represents strikes us as an important and neglected issue regardless. Therefore, if our discussion has helped highlight what is (and isn't) at issue in these debates, and inspired the pursuit of further empirical and conceptual lines of inquiry, we'd take our efforts, and those of our commentators, to have been worthwhile.

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